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Ship model calibration

Determination of a Ship Model's Moment of Inertia

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Determination of a Ship Model's Moment of Inertia

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Abstract

This report describes the determination of the inertia tensor of a ballasted ship model in the towing tank. The theoretical background is based on the principle of a physical pendulum: the oscillation period is proportional with the moment of inertia about the oscillation axis. The governing formulae are determined, including the uncertainty on each term. The practical application is tested with a generic profile. A working example is provided for the benchmark container ship KCS.

This report is basically a translation of (Delefortrie *et al.*, 2012), updated with a blind benchmark test and applied to a known ship model.

fields of knowledge: Uitrusting > Schepen > Overig

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1. Motivation

Tello Ruiz et al., 2016 mentioned the peculiar fact of a constant offset between the moment of inertia about the ship's longitudinal axis as measured on the towing tank and the value given by the numerical package Hydrostar. Therefore the pendulum theory has been completely checked. This was performed by the derivation of the moments of inertia of a generic profile, with known inertia tensor, but initially unknown to the staff of the towing tank.

At the same time the previous report (Delefortrie *et al.*, 2012) was translated to English and the benchmark vessel KCS is now used as example.

2. Ship behaviour in 6 DOF

The following set of equations expresses Newton's second law in the axis system presented in Figure 1.

$$X = m[\dot{u} - vr + wq - x_G(q^2 + r^2) + y_G(-\dot{r} + pq) + z_G(\dot{q} + pr)] \tag{1}$$

$$Y = m[\dot{v} + ur - wp + x_G(\dot{r} + pq) - y_G(p^2 + r^2) + z_G(-\dot{p} + qr)] \tag{2}$$

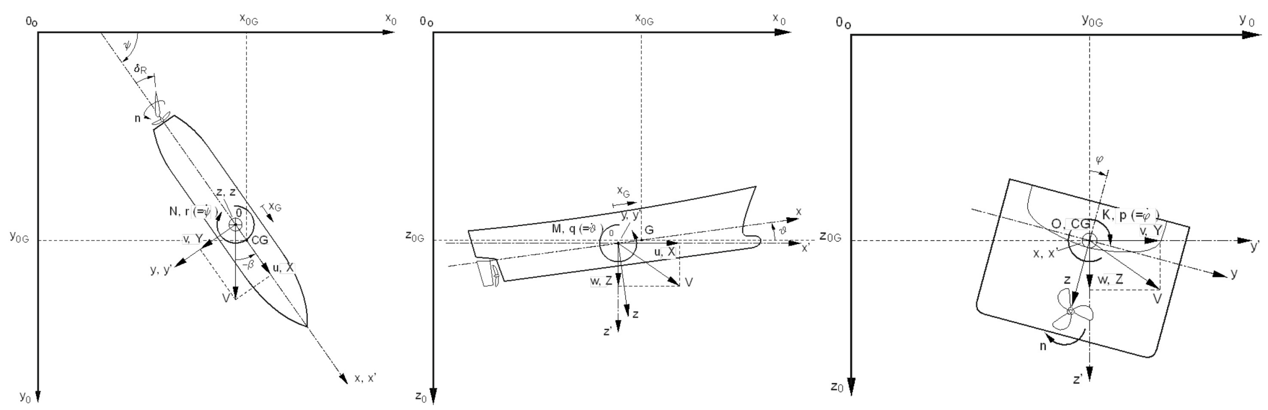
$$Z = m[\dot{w} + uq - vp + x_G(-\dot{q} + pr) + y_G(\dot{p} + qr) - z_G(p^2 + q^2)] \tag{3}$$

$$K = (I_{xx}\dot{p} - I_{xy}\dot{q} - I_{xz}\dot{r}) + (I_{yx}r - I_{xz}q)p + (I_{zz} - I_{yy})qr + I_{yz}(r^2 - q^2) + m[(\dot{w} + vp - uq)y_G - (\dot{v} + ur - wp)z_G] \tag{4}$$

$$M = (-I_{xy}\dot{p} + I_{yy}\dot{q} - I_{yz}\dot{r}) + (I_{yz}p - I_{xy}r)q + (I_{xx} - I_{zz})pr + I_{xz}(p^2 - r^2) + m[-(\dot{w} + vp - uq)x_G + (\dot{u} - vr + wq)z_G] \tag{5}$$

$$N = (-I_{xz}\dot{p} - I_{yz}\dot{q} + I_{zz}\dot{r}) + (I_{xz}q - I_{yz}p)r + (I_{yy} - I_{xx})pq + I_{xy}(q^2 - p^2) + m[(\dot{v} + ur - wp)x_G - (\dot{u} - vr + wq)y_G] \tag{6}$$

Figure 1 - Ship and earth fixed coordinate systems in 6 degrees of freedom



projections on the x_0y_0-plane, y_0z_0-plane and z_0x_0-plane.

During a captive model tests, the forces X, Y, (Z) and the moments K, (M), N are continuously measured. To be able to predict the external forces correctly,

- the ship's mass m;
- the ship's centre of gravity $\boldsymbol{r_G}$;
- the inertial tensor about the origin $\boldsymbol{I_0}$:

$$\boldsymbol{I_0} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \tag{7}$$

have to be known. The inertial components are determined based on the principle of the physical pendulum.

3. Theoretical background

3.1. Principle

The oscillation period T of a physical pendulum is determined by:

$$T = 2\pi\sqrt{\frac{I_0}{mgl_s}} \quad (8)$$

The parameters in this equation are:

- I_0: moment of inertia about the oscillation axis [kgm²];
- m: pendulum's mass [kg];
- g: gravity acceleration [m/s²];
- l_s: distance between the pendulum's centre of gravity and the oscillation axis [m].

If the oscillation period is known, the moment of inertia can be expressed as:

$$I_0 = \frac{T^2}{4\pi^2} mgl_s \quad (9)$$

Application of the parallel axis theorem (Huygens-Steiner theorem) yields the moment of inertia about a parallel axis through the centre of gravity:

$$I_G = \frac{T^2}{4\pi^2} mgl_s - ml_s^2 \quad (10)$$

The unknown parameters in this equation are:

- the oscillation period T;
- the mass m;
- the position of the centre of gravity l_s.

These parameters can be determined both for an empty and a ballasted pendulum (with and without ship model).

3.2. Determination of the mass

The mass can be determined by simply weighing the ship model and the pendulum. The measurement accuracy on the towing tank is 0.2 kg. Some components can be measured with a better accuracy, but in a conservative approach an accuracy of 0.2 kg will be maintained.

3.3. Determination of the pendulum's centre of gravity

3.3.1. Value

A right handed pendulum axis system is introduced, having its origin in the oscillation point. The z-axis is positive downwards. The x-axis lies in the longitudinal symmetry plane and the y-axis in the lateral symmetry plane. The oscillation axis can be either the x-axis or the y-axis, depending on the position of the pendulum.

An additional mass m_{BG} is eccentrically put on the pendulum with its centre of gravity at position (y_{BG}, $l_{BG} = z_{BG}$) referred to the oscillation axis, see Figure 2. The pendulum will move towards a new equilibrium position determined by a measurable angle β. This equilibrium position follows from the moment equilibrium

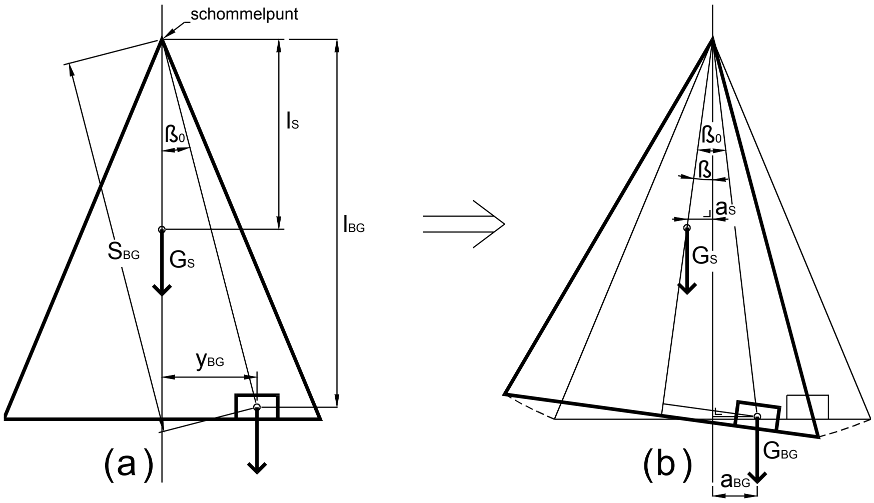
$$a_S G_s = a_{BG} G_{BG} \tag{11}$$

Both the weight of the pendulum G_s and of the additional mass G_{BG} are known values. The unknown levers a_S and a_{BG} can be expressed as a function of the angle β:

$$a_{BG} = y_{BG}\cos\beta - z_{BG}\sin\beta \tag{12}$$

$$a_S = l_s \sin\beta \tag{13}$$

Figure 2 – Ballasted pendulum



The above equations yield the position of the centre of gravity:

$$l_S = \frac{a_s}{\sin\beta} = \frac{a_{BG}G_{BG}}{G_s\sin\beta} = \frac{m_{BG}}{m_s}\left(\frac{y_{BG}}{\tan\beta} - z_{BG}\right) = z_S \tag{14}$$

3.3.2. Accuracy

The accuracy on the position of the centre of gravity depends on the accuracy of the contributing parameters in equation (14). The partial derivatives from this equation are:

$$\frac{\partial z_s}{\partial m_{BG}} = \frac{\frac{y_{BG}}{\tan\beta} - z_{BG}}{m_s} \tag{15}$$

$$\frac{\partial z_s}{\partial m_s} = -\frac{m_{BG}}{m_s^2}\left(\frac{y_{BG}}{\tan\beta} - z_{BG}\right) \tag{16}$$

$$\frac{\partial z_s}{\partial y_{BG}} = \frac{m_{BG}}{m_s\tan\beta} \tag{17}$$

$$\frac{\partial z_s}{\partial\beta} = -\frac{m_{BG}y_{BG}}{m_s\sin^2\beta} \tag{18}$$

$$\frac{\partial z_s}{\partial z_{BG}} = -\frac{m_{BG}}{m_s} \tag{19}$$

The accuracy can be improved by increasing the angle β, due to the presence of $\sin^2\beta$ in the denominator of equation (18).

3.4. Determination of the oscillation period

The oscillation period T is determined by slightly swinging the pendulum (without additional weights). The following can adversely affect the outcome:

- A too large oscillation amplitude will cause a significant damping and hence an unreliable oscillation period;
- A too small oscillation amplitude will complicate the measurement of the oscillation period.
- Badly conditioned balancing edges will increase damping.

The oscillation period should be determined both about its x-axis and y-axis. The pendulum is then referred to as X-pendulum and Y-pendulum.

3.5. Determination of the pendulum's moments of inertia

The moments of inertia are given by equation (9) or:

$$I_{XX} = \frac{T_{XX}^2}{4\pi^2} m_s g z_s = \frac{T_{XX}^2}{4\pi^2} g m_{BG} \left(\frac{y_{BG}}{\tan\beta} - z_{BG} \right) \tag{20}$$

$$I_{YY} = \frac{T_{YY}^2}{4\pi^2} m_s g z_s = \frac{T_{YY}^2}{4\pi^2} g m_{BG} \left(\frac{y_{BG}}{\tan\beta} - z_{BG} \right) \tag{21}$$

It is sufficient to determine I_{XX} and I_{YY}, because the pendulum has two perpendicular symmetry planes, which means that the products of inertia are zero.

The partial derivatives are (neglecting $z_s(m_s)$):

$$\frac{\partial I}{\partial T} = \frac{T}{2\pi^2} m_s g z_s \tag{22}$$

$$\frac{\partial I}{\partial m_s} = \frac{T^2}{4\pi^2} g z_s \tag{23}$$

$$\frac{\partial I}{\partial z_s} = \frac{T^2}{4\pi^2} g m_s \tag{24}$$

which emphasizes the importance of the accuracy of z_s, and consequently the angle β.

3.6. Determination of the ship model's centre of gravity

In this section the centre of gravity of the ship model is determined prior to ballasting. The same method as in section 3.3 could be used, but a better accuracy is achieved using the equilibrium of the pendulum + ship system. The pendulum will have a deviation of 0° when the mass is equally distributed on both sides. It is sufficient to position the ship model in such way that the pendulum' s deviation equals 0°:

- x_G: put the ship model on the Y-pendulum. The y-axis of the ship model should be parallel to the axis of the pendulum and the pendulum's deviation equals 0°. The distance between the midship section and the pendulum axis is then x_G;

- y_G: in most cases equal to 0 m. Put the ship model on the X-pendulum. The x-axis of the ship model should be parallel to the axis of the pendulum and the pendulum's deviation equals 0°. The distance between the longitudinal symmetry plane and the pendulum axis is then y_G. For most ship models y_G is equal to 0 m and small ballast is added to the ship if the port and starboard weight are different;

- z_G: put the ship model on its side on the X-pendulum. The x-axis of the ship model should be parallel to the axis of the pendulum and the pendulum's deviation equals 0°. The distance between the water plane and the pendulum axis is then z_G.

The accuracy of the position depends on the accuracy of the angle of the pendulum and the method to measure the distance. The latter is the most important and is expected to have an accuracy of 0.001 m.

3.7. Determination of the ship model's inertia

3.7.1. Moments of inertia

About the x- and y-axis

The moments of inertia can be determined according to section 3.5. For the present case the equations are:

$$I_{XXT} = \frac{T_{XXT}^2}{4\pi^2} m_\Sigma g z_\Sigma \tag{25}$$

$$I_{YYT} = \frac{T_{YYT}^2}{4\pi^2} m_\Sigma g z_\Sigma \tag{26}$$

The ship model has to be positioned on the pendulum so that the centre of gravity of both the ship model and the pendulum are on the same vertical.

- I_{XXT}: the ship model is put on the X-pendulum with the x-axis of the ship parallel to the pendulum's axis.
- I_{YYT}: the ship model is put on the Y-pendulum with the y-axis of the ship parallel to the pendulum's axis.

In the above equations:

- m_Σ: total mass of ship model and pendulum;
- z_Σ: position of the centre of gravity of the ship and pendulum (below the swinging point):

$$z_\Sigma = \frac{m_{pendulum} z_{pendulum} + m_{ship} z_{ship}}{m_\Sigma} \tag{27}$$

The moment of inertia of the ship about the pendulum's axis is found by superposition:

$$I_{schip} = I_T - I_{pendulum} \tag{28}$$

With equation (10) the moment of inertia about any axis can be found.

About the z-axis

It is also possible to determine the moment of inertia about the z-axis. If the ship model is put on its side on the Y-pendulum (the so-called Z-pendulum), the ship model is swung about its z-axis. Equation (26) is still valid, but z_{ship} in (27) has to be interpreted as the position of y_G with respect to the pendulum's axis. As in most cases y_G equals zero, this position is:

$$d - \frac{B}{2} \tag{29}$$

with d the height of the swinging point above the pendulum's plane.

3.7.2. Products of inertia

The determination of the moment of inertia about the z-axis is a specific case of the generic formula for the moment about any axis Δ:

$$I_\Delta = I_{xx}\cos^2\alpha + I_{yy}\cos^2\beta + I_{zz}\cos^2\theta - 2I_{xy}\cos\alpha\cos\beta - 2I_{yz}\cos\beta\cos\theta - 2I_{xz}\cos\alpha\cos\theta \tag{30}$$

In this equation:

- α represents the angle between the x-axis of the ship and Δ;
- β represents the angle between the y-axis of the ship and Δ;
- θ represents the angle between the z-axis of the ship and Δ;

In case the ship model is put on the X-pendulum with its x-axis parallel to the pendulum's axis:

- α equals 0 or 180°;
- β equals -90 or 90°;
- θ equals -90 or 90°;

And equation(30) leads to:

$$I_\Delta = I_{xx} \tag{31}$$

It is important to use the correct sign of the angles for the determination of the products of inertia. Delefortrie *et al.* (2009) explains how to determine the correct sign. The projection on the z_0x_0-plane in Figure 1 can also be used as a top view of the ship on the Z-pendulum.

For a conventional ship, both form and mass symmetry about the xz-plane can be assumed, so that:

$$I_{xy} \approx I_{yz} \approx 0 \tag{32}$$

Because the xz-plane is the only symmetry plane, I_{xz} should be determined. Combining (30) and (32) yields:

$$I_\Delta = I_{xx}\cos^2\alpha + I_{yy}\cos^2\beta + I_{zz}\cos^2\theta - 2I_{xz}\cos\alpha\cos\theta \tag{33}$$

A solution for I_{xz} is found provided $\cos\alpha\cos\theta$ is different from zero. The value for $\cos^2\beta$ can be freely chosen, for instance 0. This is the case when the ship is put on the Z-pendulum with an angle α (between the x-axis of the ship and the pendulum axis) different from ±90°. In any case:

$$\alpha = \theta + 90 \tag{34}$$

Equation (33) can now be written as:

$$I_\Delta = I_{xx}\cos^2\left(\theta + \frac{\pi}{2}\right) + I_{zz}\cos^2\theta - 2I_{xz}\cos\left(\theta + \frac{\pi}{2}\right)\cos\theta \tag{35}$$

resulting in:

$$I_{xz} = \frac{I_\Delta - I_{xx}\sin^2\theta - I_{zz}\cos^2\theta}{\sin 2\theta} \tag{36}$$

3.7.3. In general

Put the ship model on the pendulum in such way that the Δ-axis is parallel with the pendulum's axis (Δ-pendulum). At rest the deviation of the pendulum should be equal to zero.
The oscillation period T_Δ during a slight swing is measured, yielding:

$$I_{\Delta T} = \frac{T_{\Delta T}^2}{4\pi^2} m_\Sigma g z_\Sigma \tag{37}$$

$m_\Sigma g z_\Sigma$ is determined with equation (27), in which z_{ship} represents the position of the centre of gravity of the ship below the swinging point. It is possible to use the set of equations (22) – (24), but both for the pendulum and the ship model.

Equation (28) gives then the moment of inertia of the ship without pendulum. $I_{pendulum}$corresponds to I_{XX} (x-axis is the pendulum's axis) or I_{YY} (y-axis is the pendulum's axis). The partial derivatives of these equations are:

$$\frac{\partial I_{ship}}{\partial I_T} = 1 \tag{38}$$

$$\frac{\partial I_{ship}}{\partial I_{pendulum}} = -1 \tag{39}$$

Steiner's theorem allows to find the inertia about the axes through the centre of gravity:

$$I_G = I_0 - m_{ship} z_{ship}^2 \tag{40}$$

with partial derivatives:

$$\frac{\partial I_G}{\partial m_{ship}} = -z^2{}_{ship} \tag{41}$$

$$\frac{\partial I_G}{\partial z_{ship}} = -2m_{ship} \tag{42}$$

In case of the product of inertia I_{xz}, equation (36) has to be solved as well, with partial derivatives:

$$\frac{\partial I_{XZ}}{\partial I_\Delta} = \frac{1}{sin(2\theta)} \tag{43}$$

$$\frac{\partial I_{XZ}}{\partial I_{XX}} = -\frac{1}{2}\tan\theta \tag{44}$$

$$\frac{\partial I_{XZ}}{\partial I_{ZZ}} = -\frac{1}{2} cot\,\theta \tag{45}$$

$$\frac{\partial I_{XZ}}{\partial \theta} = -\frac{1}{2}\left[\frac{1}{sin^2\theta} - \frac{1}{cos^2\theta}\right] I_\Delta - \frac{1}{2}\frac{1}{cos^2\theta} I_{XX} + \frac{1}{2}\frac{1}{sin^2\theta} I_{ZZ} \tag{46}$$

The last equation is the most significant one, the angle θ should not be chosen too small.

3.8. Adding ballast

Once the position of the centre of gravity and the inertia tensor are known, ballast weights are added to the ship to achieve a certain loading condition. By adding ballast weights:

- the resulting weight corresponds to the displacement of the ship at the loading condition;
- the centre of gravity is on the same vertical as the centre of buoyancy of the loading condition;
- the initial stability levers have their desired value;
- the moments of inertia have realistic values, which are mostly expressed as a function of the dimensions and the ship's mass.

The ballast weights have their own inertia. Adding any weight changes:

- the total mass, equation (47);
- the position of the centre of gravity, equation (48);
- the inertia tensor, equation (59).

3.9. Superposition and convoys

The above reasoning is also applicable to convoys. Suppose a convoy is made of n parts, each part i with its own centre of gravity x_{Gi}, y_{Gi} and z_{Gi} in its own axis system, its mass m_i and its inertia tensor $\boldsymbol{I_{Gi}}$ about its centre of gravity. In this case:

$$m = \sum_{i=0}^{n} m_i \tag{47}$$

represents the total mass of the convoy and

$$r_G = \frac{\sum_{i=0}^{n} m_i r_{Gi}}{m} \tag{48}$$

the position of the centre of gravity of the convoy, with r_{Gi} ($r = x, y, z$) the position of the centre of gravity expressed in the convoy axis system.

Summation of the inertia of each convoy part is possible:

$$\boldsymbol{I_0} = \sum_{i=0}^{n} \boldsymbol{I_{0i}} \tag{49}$$

provided the inertia tensor of each part is expressed in the convoy axis system. This is achieved with Steiner's theorem, possibly with previous rotations:

- For the moments of inertia, for instance about the x-as, a translation from G(0,0,0) to O(x, y, z):

$$I_{XX0} = I_{XXG} + m(y^2 + z^2) \tag{50}$$

- For the products of inertia, for instance I_{XZ}, a translation from G(0,0,0) to O(x, y, z):

$$I_{XZ0} = I_{XZG} + mxz \tag{51}$$

A rotation may be needed when a barge is put backwards in a convoy. In this case:

$$I_\Delta = I'_\Delta = I'_{xx}\cos^2(\alpha+\pi) + I'_{yy}\cos^2(\beta+\pi) + I'_{zz}\cos^2\theta - 2I'_{xy}\cos(\alpha+\pi)\cos(\beta+\pi) - 2I'_{yz}\cos(\beta+\pi)\cos\theta - 2I'_{xz}\cos(\alpha+\pi)\cos\theta \tag{52}$$

which yields:

$$I_\Delta = I'_\Delta = I'_{xx}\cos^2\alpha + I'_{yy}\cos^2\beta + I'_{zz}\cos^2\theta - 2I'_{xy}\cos\alpha\cos\beta + 2I'_{yz}\cos\beta\cos\theta + 2I'_{xz}\cos\alpha\cos\theta \tag{53}$$

Comparing (53) and (30) shows that in this case I_{yz} and I_{xz} need to swap sign.

4. Blind calibration with a generic profile

4.1. Overview

A generic profile (manufactured by Rose Krieger) with the characteristics mentioned in Table 1 was put on the medium size pendulum (January 3, 2017) to determine its moments of inertia.

Table 1 – Characteristics of the generic profile

Measured on towing tank: origin of the axis system in the middle of the volume. The centre of gravity corresponds with the origin of the axis system.	
Weight	111.72 kg
Length	2.9995 m
Width	0.320 m
Depth	0.160 m
Moments of inertia provided by manufacturer (disclosed after the measurements)	
I_{XX}	1.684 kgm²
I_{YY}	84.141 kgm²
I_{ZZ}	85.065 kgm²

The characteristics of the medium size pendulum are (calibration of June 21, 2012):

- $m_{pendulum}$= 50.8 kg;
- Equation (14): $z_S = 0.296 \pm 0.004\ m$;
- Equation (20): $I_{XX} = 17.8 \pm 0.1\ kgm^2$;
- Equation (21): $I_{YY} = 19.3 \pm 0.1\ kgm^2$;

The distance between the base of the medium size pendulum and its swinging point is 0.875 m.

4.2. Determination of the moment of inertia about x-axis

The focus is put here on the moment of inertia about the x-axis, because its deviation was the main reason to perform the present calibration. The pendulum with the generic profile was slightly swung 10 times and the time for five oscillation periods was registered.

Table 2 – Measurement of the oscillation period of the medium size pendulum with generic profile

Measurement	$5T$ (s)	I_{XXT} (kgm²) (eq. 25)	Measurement	$5T$ (s)	I_{XXT} (kgm²) (eq. 25)
1	9.47	92.574907	6	9.48	92.77052222
2	9.48	92.770522	7	9.47	92.57490705
3	9.48	92.770522	8	9.47	92.57490705
4	9.48	92.770522	9	9.47	92.57490705
5	9.48	92.770522	10	9.48	92.77052222

To evaluate equation (25) the following values were used:

- Position of the centre of gravity of the profile: $0.875 - \frac{0.16}{2} = 0.795\ m$;
- Total mass: 50.8 + 111.72 = 162.52 kg.

Of course, the result of equation (25) in Table 2 has too many significant digits. The partial derivatives are the extension of the set of equations (22)-(24):

$$q_1 = \frac{\partial I}{\partial T} = \frac{gT}{2\pi^2}\left[m_{ship}z_{ship} + m_{pendulum}z_{pendulum}\right] \tag{54}$$

$$q_2 = \frac{\partial I}{\partial m_{ship}} = \frac{gT^2}{4\pi^2}z_{ship} \tag{55}$$

$$q_3 = \frac{\partial I}{\partial m_{pendulum}} = \frac{gT^2}{4\pi^2}z_{pendulum} \tag{56}$$

$$q_4 = \frac{\partial I}{\partial z_{ship}} = \frac{gT^2}{4\pi^2}m_{ship} \tag{57}$$

$$q_5 = \frac{\partial I}{\partial z_{pendulum}} = \frac{gT^2}{4\pi^2}m_{pendulum} \tag{58}$$

The uncertainties mentioned Table 3 in are applicable on the single measurements.

Table 3 – Uncertainties on each parameter

Parameter	Uncertainty	Explanation
∂T	0.01 s	Resolution chronometer
∂m_{ship}	0.2 kg	Resolution balance
$\partial m_{pendulum}$	0.2 kg (0.01 kg)	Resolution bridge balance (resolution second balance)
∂z_{ship}	0.0015 m (0.001 m)	Resolution distance measurement
$\partial z_{pendulum}$	0.002 m	Half of the uncertainty of the calibration of June 21, 2012

The expanded uncertainty is then, for a single measurement:

$$\partial I = \sqrt{\left[(q_1 \partial T)^2 + \left(q_2 \partial m_{ship}\right)^2 + \left(q_3 \partial m_{pendulum}\right)^2 + \left(q_4 \partial z_{ship}\right)^2 + \left(q_5 \partial z_{pendulum}\right)^2\right]} \tag{59}$$

The weight w_i of each oscillation period measurement i is equal to:

$$w_i = \left(\frac{1}{\partial I_i}\right)^2 \tag{60}$$

The average moment of inertia of the 10 measurements is:

$$I = \sum \frac{w_i I_i}{w_i} \tag{61}$$

With corresponding uncertainty

$$\partial I = \frac{1}{\sqrt{\sum w_i}} \tag{62}$$

In this case:

$$I_{XXT} = 92.7 \ \pm 0.3 \ kgm^2 \text{ (68\% confidence)} \tag{63}$$

Using equations (28) and (10), the moment of inertia about the longitudinal axis of the generic profile is 92.7 – 17.8 – 111.72 x 0.795² = 4.3. The partial derivatives can be computed with equations (38) – (42). The 95% confidence is 1 kgm², thus for the generic profile:

$$I_{XX} = 4 \ \pm 1 \ kgm^2 \text{ (95\% confidence)} \tag{64}$$

The obtained moment of inertia does not agree with the theoretical value mentioned in Table 1.

The above methodology can also be applied for the moments of inertia about y- and z-axis, the resulting values are:

$$I_{YY} = 82 \ \pm 1 \ kgm^2 \text{ (95\% confidence)} \tag{65}$$

$$I_{ZZ} = 82 \ \pm 1 \ kgm^2 \text{ (95\% confidence)} \tag{66}$$

In other words, none of the moments of inertia is predicted well enough.

4.3. Problem identification and remediation

The calibration of the generic profile seems to confirm the findings of Tello Ruiz et al., 2016. Moreover, an offset is present for all moments of inertia, however it is clearly discernible for the moment about the longitudinal axis due to its smaller value compared to the other moments.

In the first place all formulae in the Excel sheets have been double checked, however, without finding any errors.

The present pendulum was built in 2011 and was tested in September – November 2011 by the student Steve Claes from TU Delft in the frame of his internship, however, FHR never received the report from his internship, as he decided to stop his studies. On the other hand, the written notes are still available and show the successful calibration of the *short* pendulum with different I and H – profiles.

Initially, only the short pendulum has been used. However, for bigger ship models, the medium pendulum (from 2012) and long pendulum (from 2013) were used as well. All pendulums weigh 50.8 kg, the only difference is the position from the base referred to the pendulum's axis. Table 4 summarises the different calibrations. In addition, only the short pendulum was tested with known profiles. The other calibrations were performed with the same spreadsheet, but due to the long interval between the different calibrations, the results have never been compared before.

Table 4 – Calibrations of the three pendulums

Parameter	Short	Medium	Long
Calibration date	December 1, 2011	June 21, 2012	August 14, 2013
Length between base and swing point (m)	0.675 ± 0.001	0.875 ± 0.001	1.075 ± 0.001
Position of the centre of gravity below swing point (m)	0.342 ± 0.003	*0.296 ± 0.004*	*0.362 ± 0.005*
I_{XX} (kg m²)	19.1 ± 0.1	17.8 ± 0.1	24.2 ± 0.2
I_{YY} (kgm²)	21.3 ± 0.1	19.3 ± 0.1	25.6 ± 0.2

It is most noticeable, that the position of the centre of gravity of the medium pendulum is closer to the swing point compared to the short pendulum, although the base of the medium pendulum is further away. A second observation, is that the position of the centre of gravity of the long pendulum is somewhat lower, but not sufficiently lower compared to the position of its base. Mind that the evolution of the moments of inertia is also strange, but this is ascribed to the influence of the centre of gravity.

The only explanation for this can be that the centres of gravity of the medium and long pendulum are wrong. As the same spreadsheet was used for all pendulums, the only error could have occurred with the input data. Considering equation (14) and the fact that in all cases the same calibration weight was used, the error must be ascribed to the position of the calibration weight (y_{BG}, z_{BG}) and/or the reading of the deviation angle β, and most probably to y_{BG} as z_{BG} is computed automatically depending on the pendulum's size. Please observe in Table 5 the different input position for y_{BG} for the short pendulum, compared to the medium and long pendulum.

Table 5 – Position of the calibration weight

Parameter	Short	Medium	Long
Calibration date	December 1, 2011	June 21, 2012	August 14, 2013
y_{BG} (m)	0.400 ± 0.001	0.300 ± 0.001	0.300 ± 0.001

Suppose y_{BG} of the short pendulum is the true value for all pendulums. In this case the results of the medium pendulum are:

- Equation (14): $z_S = 0.448 \pm 0.005\ m$;
- Equation (20): $I_{XX} = 27.0 \pm 0.2\ kgm^2$;
- Equation (21): $I_{YY} = 29.3 \pm 0.2\ kgm^2$;

And the results of the calibration of the generic profile would be:

$$I_{XX} = 1.97\ \pm 1\ kgm^2 \text{ (95\% confidence)} \tag{67}$$

$$I_{YY} = 84.67\ \pm 1\ kgm^2 \text{ (95\% confidence)} \tag{68}$$

$$I_{ZZ} = 85.41\ \pm 1\ kgm^2 \text{ (95\% confidence)} \tag{69}$$

On purpose, more digits were added to the average value. The differences with the theoretical values are respectively 0.32, 0.86 and 0.07 kgm², which all are in the uncertainty interval.

Clearly the wrong value of y_{BG} has been used for the medium and long pendulum. On the pendulum, marks were put to indicate were to place the ballast weight (a disc of diameter 0.2 m). The disc was placed on the correct mark, but the distance of the mark (border of the disc) was written down, instead of the distance of the centre of the disc.

Fixing this error leads to a decrease of 2.5 kgm² of I_{XX} for C0W07 in Tello Ruiz et al., 2016 or a fraction 0.73, which corresponds to the k-factor introduced in Tello Ruiz et al., 2016.

5. New determination of the pendulum's inertia

5.1. Overview

From January 4 to January 6, 2017 the pendulum has been calibrated again in the three positions. Each position was tested with the generic profile. In the following paragraphs will the calibration will be discussed for each position. The calibration of the pendulum is performed according to sections 3.2 to 3.5. The mass of the pendulum is always 50.8 ± 0.2 kg.

5.2. Medium pendulum

5.2.1. Position of the centre of gravity

The position of the centre of gravity has been determined by multiple measurements on both the X- and Y-pendulum, however only the most eccentric positions of the ballast disc are considered, due to the larger swing angle. The parameters of equation (14) are:

- For all pendulums:
  - m_S = 50.8 ± 0.2 kg;
  - m_{BG} = 9.899 ± 0.001 kg;
  - y_{BG} = ± 0.400 ± 0.001 m.
- For the medium pendulum:
  - z_{BG} = 0.8555 ± 0.001 m
- For the medium X-pendulum:
  - β_0 = -0.09° ± 0.05 ° (the measured offset angle)
  - β = 7.14° ± 0.05 ° (positive y_{BG}) and -7.34° ± 0.05 ° (negative y_{BG})
- For the medium Y-pendulum:
  - β_0 = -0.30° ± 0.05 ° (the measured offset angle)
  - β = 6.94° ± 0.05 ° (positive y_{BG}) and -7.62° ± 0.05 ° (negative y_{BG})

The net angles for the four measurements are (β-β_0) listed in Table 6.

Table 6 – Determination of the centre of gravity of the medium pendulum

Measurement	I	II	III	IV
Medium pendulum	X	X	Y	Y
y_{BG} (m)	0.400 ± 0.001	-0.400 ± 0.001	0.400 ± 0.001	-0.400 ± 0.001
β-β_0 (rad)	0.1262 ± 0.0009	0.1265 ± 0.0009	0.1264 ± 0.0009	0.1278 ± 0.0009
z_S (m) (equation 14)	0.4472	0.4455	0.4464	0.4396

The expanded uncertainty on the centre of gravity can be computed with the partial derivatives of equation (14). Following a similar method as described in equations (59) – (62), the following average value is obtained:

$$z_S = 0.445 \pm 0.005\ m \text{ (95\% confidence)} \tag{70}$$

5.2.2. Moments of inertia

The moments of inertia are determined based on the oscillation periods mentioned in Table 7 and Table 8. The results of these tables can be post-processed similarly to equations (59) – (62), yielding the following values:

$$\mathrm{I_{XX}} = 26.8 \pm 0.2\ \mathrm{kgm^2} \text{ (95\% confidence)} \tag{71}$$

$$I_{YY} = 29.0 \pm 0.2\ kgm^2 \text{ (95\% confidence)} \tag{72}$$

Please observe that the values of the new calibration overlap with the remediation in section 4.3.

Table 7 – Measurement of the oscillation period of the empty medium size X-pendulum

Measurement	$5T$ (s)	I_{XX} (kgm²) (eq. 20)	Measurement	$5T$ (s)	I_{XX} (kgm²) (eq. 20)
1	10.91	26.72346	6	10.92	26.77246855
2	10.92	26.77247	7	10.92	26.77246855
3	10.91	26.72346	8	10.92	26.77246855
4	10.92	26.77247	9	10.92	26.77246855
5	10.92	26.77247	10	10.92	26.77246855

Table 8 – Measurement of the oscillation period of the empty medium size Y-pendulum

Measurement	$5T$ (s)	I_{YY} (kgm²) (eq. 21)	Measurement	$5T$ (s)	I_{YY} (kgm²) (eq. 21)
1	11.37	29.02445	6	11.37	29.02445474
2	11.36	28.97342	7	11.36	28.97342274
3	11.36	28.97342	8	11.36	28.97342274
4	11.36	28.97342	9	11.37	29.02445474
5	11.37	29.02445	10	11.36	28.97342274

5.2.3. Generic profile

The values for the profile with the medium pendulum are now:

$$I_{XX} = 2.04 \pm 1\ kgm^2 \text{ (95\% confidence)} \tag{73}$$

$$I_{YY} = 84.72 \pm 1\ kgm^2 \text{ (95\% confidence)} \tag{74}$$

$$I_{ZZ} = 85.46 \pm 1\ kgm^2 \text{ (95\% confidence)} \tag{75}$$

5.3. All pendulums

The short and long pendulum were calibrated following the same methodology. The results of all pendulums are given in Table 9.

Table 9 – New calibrations of the three pendulums

Parameter	Short	Medium	Long
Calibration date	January 6, 2017	January 4, 2017	January 5, 2017
Length between base and swing point (m)	0.675 ± 0.001	0.875 ± 0.001	1.075 ± 0.001
Position of the centre of gravity below swing point (m)	0.342 ± 0.003	**0.445 ± 0.005**	**0.548 ± 0.007**
I_{XX} (kg m²)	19.1 ± 0.1	**26.8 ± 0.2**	**38.9 ± 0.3**
I_{YY} (kgm²)	21.**2** ± 0.1	**29.0 ± 0.2**	**36.7 ± 0.3**

The calibration of the generic profile is summarised in Table 10. In all cases the moments of inertia are predicted within the uncertainty interval, however, due to the smaller uncertainties, the best predictions are achieved with the shortest pendulum. The average values tend to be over predicted with increasing pendulum length.

Table 10 – Generic profile

Parameter	Theoretical	Short	Medium	Long
Calibration date	-	January 6, 2017	January 3, 2017	January 5, 2017
I_{XX} (kg m²)	1.684	1.89 ± 0.9 (+12%)	2.04 ± 1 (+21%)	2.13 ± 1 (+26%)
I_{YY} (kgm²)	84.141	84.20 ± 1.0 (+0%)	84.72 ± 1 (+1%)	84.74 ± 2 (+1%)
I_{ZZ} (kgm²)	85.065	85.09 ± 0.9 (+0%)	85.46 ± 1 (+0%)	85.64 ± 1 (+1%)

6. Example for the KCS

6.1. Determination of the characteristics of the empty ship model[1]

The empty ship model is defined as the fully instrumented ship model with centre of gravity in the xz-plane, prior to adding ballast weights. The main dimensions of the empty KCS are:

- L_{PP} = 4.367 ± 0.001 m;
- B = 0.611 ± 0.001 m;
- T = 0.205 ± 0.001 m;
- m = 153.0 ± 0.2 kg.

6.1.1. Determination of the centre of gravity

The centre of gravity is determined according to section 3.6. The medium pendulum has been used. The initial uncertainty is estimated at 0.001 m. The final uncertainty is however large, because the measured distances depend on:

- The reading of the swing angle (should be 0°, or at least equal to the offset position at rest);
- The symmetry of the ship's position on the pendulum, which is determined by the distance between the ship model and the pendulum at different positions.

For the empty KCS the position of the centre of gravity is:

- x_G = -0.364 ± 0.002 m;
- y_G = ± 0.002 m;
- z_G = 0.0296 ± 0.003 m;

with a confidence of 95%.

6.1.2. Determination of the inertia tensor

General

The components of the inertia tensor are determined according to section 3.7. The ship model is put on the pendulum in such way that the masses are equally distributed (0° swing angle). The pendulum is swung and the duration to execute 5 oscillations is measured 10 times in the following pendulum positions:

- X-pendulum;
- Y-pendulum;
- Z-pendulum;
- Δ-pendulum (twice, with a different angle between the ship model and the pendulum's axis).

[1] http://wlsow.vlaanderen.be/shpgenerator/Lists/Modelvoorbereidingen/20170104_traagh_C0401.xlsm

X-, Y- and Z-pendulum

The determination of I_{XX}, I_{YY} and I_{ZZ} is similar. The ship model has to be positioned correctly on the pendulum (the pendulum's axis has to be parallel with the inertia axis and the centre of gravity of the ship has to be on the same vertical as the centre of gravity of the pendulum) Equations (25) and (26) yield:

$$I_{XXT} = 110.0 \ \pm 0.4\, kgm^2 \tag{76}$$

$$I_{YYT} = 345.7 \ \pm 0.7\, kgm^2 \tag{77}$$

$$I_{ZZT} = 324.0 \ \pm 0.7\, kgm^2 \tag{78}$$

The confidence is 68% (preliminary result). The moments of inertia of the ship model are computed using equation (40):

$$I_{XX} = 8 \ \pm 1\, kgm^2 \tag{79}$$

$$I_{YY} = 242 \ \pm 2\, kgm^2 \tag{80}$$

$$I_{YY} = 245 \ \pm 2\, kgm^2 \tag{81}$$

with a confidence of 95%.

Δ-pendulum

The determination of I_{XZ} is more complicated. The ship model is put with a certain angle on the Z-pendulum. The angle is determined by the distances between the keel plane the and the vertical profiles of the pendulum. The uncertainty of the measured angle is at least 0.05°. For the KCS the following two angles have been applied:

- I: -24.8037°;
- II: 23.440177°.

The resulting total moments of inertia are:

$$I_{\Delta IT} = 286.4 \ \pm 0.6\, kgm^2 \tag{82}$$

$$I_{\Delta IIT} = 281.1 \ \pm 0.6\, kgm^2 \tag{83}$$

The confidence is 68% (preliminary result). The moments of inertia of the ship model are computed using equation (40):

$$I_{\Delta I} = 207.8 \ \pm 0.7\, kgm^2 \tag{84}$$

$$I_{\Delta II} = 202.5 \ \pm 0.7\, kgm^2 \tag{85}$$

Via (36) and (43) - (46) I_{XZ} can be determined:

$$I_{XZ} = -6 \ \pm 7\, kgm^2 \tag{86}$$

with a confidence of 95%. I_{XZ} does not seem to differ significantly from zero, which is due to equation (46), which is minimal for θ = 45.

However there are reasons to still consider (86). The values for I_{XZ} which were found for the different angles θ are:

- I: -5.4 kgm²;
- II: -7.3 kgm².

The standard deviation of these two measurements is 1.3 kgm², which is significantly smaller than 7 kgm².

6.2. Ballasting the ship model[2]

6.2.1. Desired values

The ship model needs to have an even keel, with draft of 0.2051 m. This corresponds to:

- x_G = -0.065 m;
- m_{schip} = 356.2 kg.

The desired initial stability lever $\overline{GM}$ should be equal to 0.011 m, which corresponds to a vertical position of the centre of gravity of z_G = -0.068 m. This value cannot be obtained, due to the low position of the centre of gravity of the instrumented ship and the relatively low ballast weight to be added, see the previous section. Instead z_G will be equal to -0.030 m, yielding $\overline{GM}$ = 0.0487 m.

The desired moments of inertia are:

$$I_{XX} = m_{schip}(0.35B)^2 = 16.3\ kgm^2 \quad (87)$$

$$I_{YY} = I_{ZZ} = m_{schip}(0.25L_{PP})^2 = 424.6\ kgm^2 \quad (88)$$

There are no specific requirements for I_{XY}. If the other requirements are met, I_{XY} should be as close as possible to zero. The uncertainties on the connection mechanism and the ballast weights is significantly smaller compared to the ship model.

6.2.2. Connection plates

Connection plates need to be added to support the gimbal. The U-shaped beam, the roll motor and the connection plates add to the ship's displacement. They can also add to the moments of inertia:

- Never about the y-axis (always free to pitch);
- Mostly about the x-axis (roll is mostly fixed);
- Always about the z-axis (yaw is always fixed).

If a hinge is present, all mass above this hinge will not add to the moment of inertia or centre of gravity of the ship model. The KCS will only be free to pitch (and heave), thus only I_{XX}[3] and I_{ZZ} of the beam will be added.

The share of the beam, roll motor and connection plates is:

- m_{beam} = 78.551 kg;
- $I_{XX\ beam}$ = 0.5024 kgm²
- $I_{YY\ beam}$ = 0.4784 kgm²
- $I_{ZZ\ beam}$ = 13.0816 kgm²
- $I_{XZ\ beam}$ = 0.0 kgm²

[2] http://wlsow.vlaanderen.be/shpgenerator/Lists/Modelvoorbereidingen/20170109_ballast_C0401.xlsm
[3] The moment of inertia about the x-axis of the U-shape is considered to be negligible. I_{XX} is thus not adapted.

expressed about the origin of the ship fixed axis system. To add the inertia of the empty ship model, they must be expressed about the same origin, using equations (50) and (51). The system ship + beam has then the following values (68% confidence):

- x_G = -0.241 ± 0.001 m;
- z_G = 0.028 ± 0.002 m;
- m_{schip} = 231.6 ± 0.2 kg;
- I_{XX} = 8.6 ± 0.5 kgm²;
- I_{YY} = 263 ± 1 kgm²;
- I_{ZZ} = 278 ± 1 kgm²;
- I_{XZ} = -8 ± 4 kgm².

6.2.3. Determination of the position of the ballast weights

The loading condition of the empty ship model with the beam, roll motors and connection plates does not correspond with the desired one. Ballast weights have to be added to obtain the desired loading condition. This is a rather iterative process, which starts by adding the heaviest ballast weights. Adding a ballast weight affects the inertia distribution as expressed by equations (47) to (49).

The user provides the available coordinates (of the centre of gravity of the ballast weights) where ballast weights can be added. A macro tests all possible ballast distributions and selects the distribution which minimizes a cost function. This cost function is based on the differences between the actual inertia and the desired inertia. After adding the ballast weights, and after checking the ship's draft, the following inertia is obtained:

$$\boldsymbol{r_G} = \begin{bmatrix} -0.070 \pm 0.002 \\ 0 \\ -0.030 \pm 0.003 \end{bmatrix} \text{m} \tag{89}$$

$$\boldsymbol{I_0} = \begin{bmatrix} 13.9 \pm 1 & 0 & -10.8 \pm 7 \\ 0 & 406.9 \pm 2 & 0 \\ -10.8 \pm 7 & 0 & 422.6 \pm 2 \end{bmatrix} \text{kgm}^2 \tag{90}$$

The obtained mass is 354.4 $\pm$ 0.2 kg. Mind the differences with the theoretical values.

6.2.4. Checking the vertical position of the centre of gravity

The vertical position of the centre of gravity has been checked by performing two heel tests. In a heel test the relationship

$$K = \Delta \overline{GM} \varphi \tag{91}$$

is tested for small heel angles.

In a first test (free heel), the ship was free to heel, and calibrated weights were added on each side of the ship. With an inclinometer the resulting heel angle was measured.

In a second test (forced heel), the roll engine was used to set the ship at a certain heel angle. The corresponding roll moment was measured.

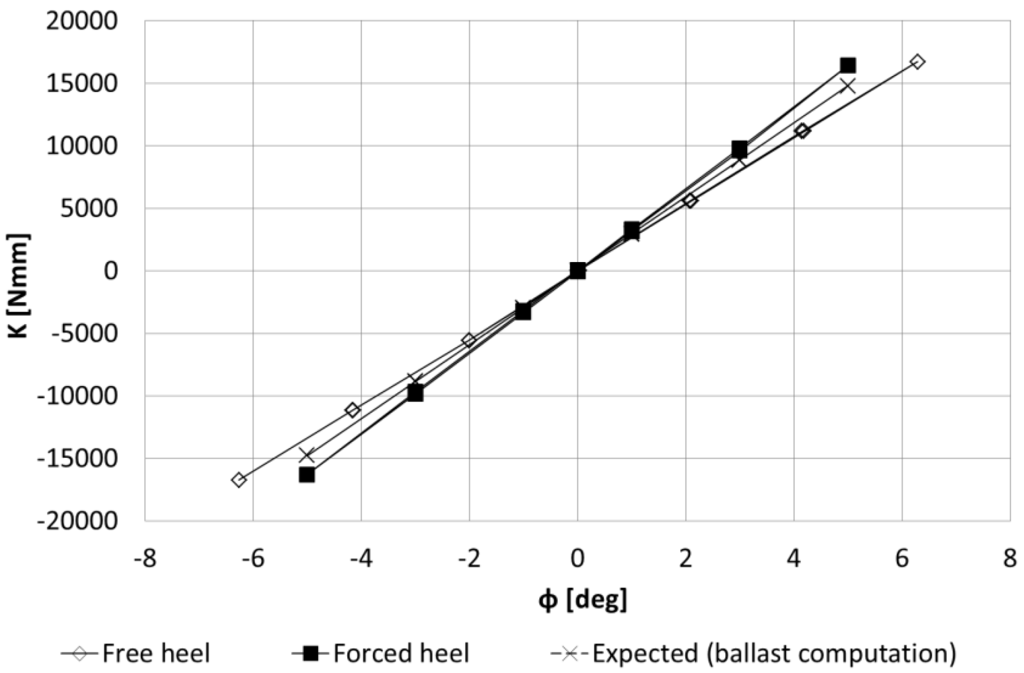
Figure 3 shows the corresponding relationships. The resulting vertical positions of the centre of gravity are:

$$z_{G,free} = -0.034\, m \tag{92}$$

$$z_{G,forced} = -0.025\, m \tag{93}$$

The average value of both measurements is equal to the expected value, but the deviation is larger than the expected one.

Figure 3 – Relationship between roll moment and heel angle



7. Recommendations

- Use the shortest possible pendulum;
- Ensure that pendulum edges are clean and sharp;
- Any change to the pendulum should be validated with a new calibration for all pendulum positions, including the determination of the inertia of a known profile;
- Determination of angles induce the largest uncertainty and should be avoided whenever possible. Although the inclinometer has a resolution of 0.05°, significant uncertainties are introduced due to the term ${}^{1}\!/_{\sin^2\alpha}$ in the partial derivative. If an angle has to be measured its value should be as large as possible (→45°) and/or the uncertainty has to be decreased by a significant number of repeat measurements;
- Due to the uncertainty, it is irrelevant to use more than 1 decimal in the value of the moments of inertia;
- Probably the uncertainties on the position of the centre of gravity are larger than the computed ones;
- Inertia computations which were carried out between 2012 and 2016 can be corrected, but this will only be done for specific cases (e.g. benchmark data).
- This report replaces (Delefortrie *et al.*, 2012).

8. References

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