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Geïdealiseerde processtudie van systeemovergangen naar hypertroebelheid

WP 1.3 Basismodel getij en zout

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Geïdealiseerde processtudie van systeemovertgang naar hypertroebheid

WP 1.3 Basismodel getij en zout

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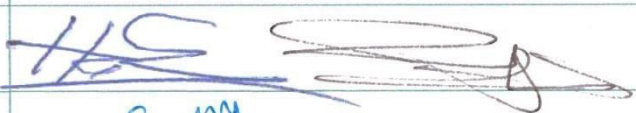
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Abstract

In dit rapport wordt een geïdealiseerd hydrodynamisch model beschreven dat de breedtegemiddelde waterbeweging onder invloed van getij, rivierafvoer en zoutverdeling beschrijft. Het model wordt toegepast op het Schelde-estuarium tussen Vlissingen en Gentbrugge, waarbij calibratie geschiedt op basis van het verticale M_2 -en M_4 -getij. Het blijkt dat de amplitude en fase van deze componenten redelijk goed wordt beschreven, en ook de waargenomen asymmetrie van de waterstand wordt kwalitatief gereproduceerd. De harmonische componenten M_2 en M_4 van gemeten cross-sectioneel gemiddelde snelheid worden gebruikt ter validatie van het model. Hierbij blijken de waargenomen en berekende amplitudes qua orde van grootte overeen te komen, maar qua langsverloop is er minder overeenstemming. De horizontale getij-asymmetrie komt goed overeen voor de Westerschelde, maar niet voor de Zeeschelde waar metingen een grote spreiding laten zien. Er wordt beargumenteerd dat deze matige overeenkomst tussen model en waarnemingen teruggevoerd kan worden op de te korte lengte van de gebruikte tijdreeks voor snelheidsmetingen. Voor de verdeling van de reststroming wordt gevonden dat deze bij hoge afvoeren door de rivierstroming wordt gedomineerd, terwijl bij lage afvoeren de Stokes terugkeerstroming de grootste bijdrage levert. De gravitationele circulatie geeft een geringe bijdrage aan de reststroming en lijkt van onderschikt belang.

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1 Inleiding

Dit rapport betreft de oplevering van deliverable 1.3 van WL-project 13_103 "Geïdealiseerde processtudie van systeemovergangen naar hypertroebelheid". Dit rapport beschrijft het hydrodynamische gedeelte van het geïdealiseerde model dat zal worden gebruikt om later in het project de overgang van lage naar hoge hypertroebelheid te bestuderen. Naast de waterbeweging door getij zijn ook effecten van zoutverdeling op de stroming in dit model verdisconteerd.

Omdat de feitelijke beschrijving vrij technisch is, is besloten om de globale opzet en resultaten afzonderlijk samen te vatten in Hoofdstuk 2. In de daarop volgende hoofdstukken 3 tot en met 6 worden respectievelijk de modelopzet de gebruikte meetgegevens, de calibratieprocedure en de modelresultaten in meer detail in het Engels beschreven. De wiskundige details van de benaderingen die aan het model ten grondslag liggen, alsmede de gebruikte oplossingsmethode, worden separaat in Appendix A1 behandeld.

2 Samenvatting

2.1 Modelopzet

2.1.1 Modelgeometrie

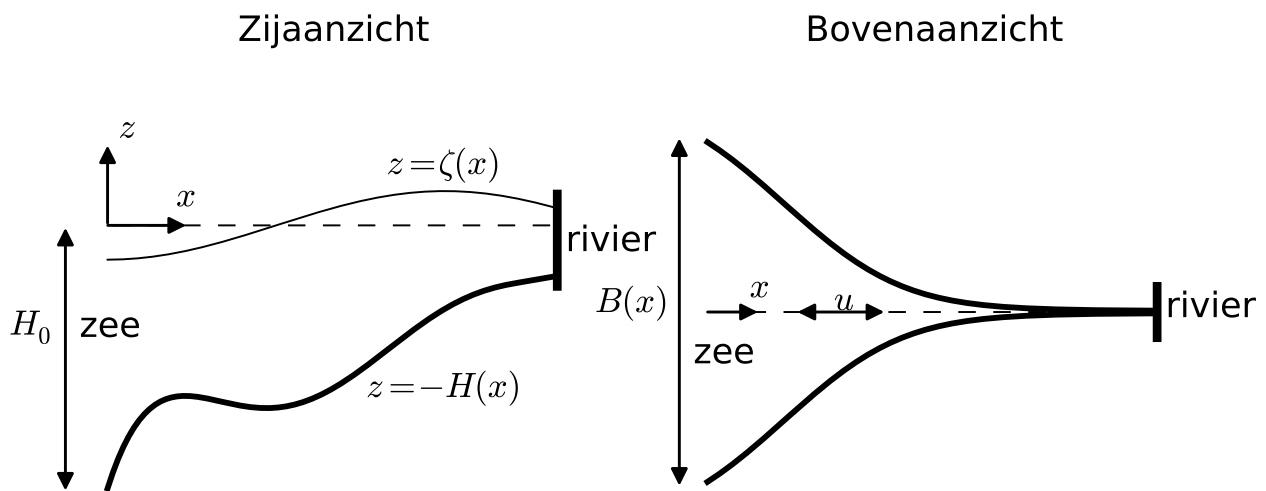
Het model is gebaseerd op het werk van Ianniello, (1977), Ianniello, (1979) en Chernetsky *e.a.*, (2010) en beschrijft de breedtegemiddelde waterbeweging in een getijkanaal van eindige lengte L . Hierbij wordt met "waterbeweging" feitelijk de breedtegemiddelde horizontale snelheid $u(x, z, t)$, de breedtegemiddelde verticale snelheid $w(x, z, t)$ en de breedtegemiddelde waterstand $\zeta(x, t)$ bedoeld. Het zeegat is gesitueerd op $x = 0$ waar een harmonische waterstand wordt voorgeschreven die enkel een M_2 -en M_4 -component bevat, ergo

$$\zeta(0, t) = A_{M_2} \cos(\sigma t) + A_{M_4} \cos(2\sigma t - \varphi), \quad (1)$$

Hierbij zijn A_{M_2} en A_{M_4} de amplitude van de M_2 -en M_4 -waterstand op de rand en is φ het bijbehorende faseverschil tussen beide getijcomponenten. De grootte σ is de cirkelfrequentie van het M_2 -getij (periode $T \approx 12.4$ uur) en is gegeven door $\sigma = 2\pi/T \approx 1.41 \times 10^{-4} \text{ rad s}^{-1}$.

Op de landwaartse rand ($x = L$) is een stuw gelokaliseerd die een constante rivierafvoer Q onderhoudt. De verticale positie z is opwaarts positief georiënteerd, waarbij $z = 0$ samenvalt met het gemiddelde zeeniveau. De breedtevariatie wordt aangegeven als $B(x)$ terwijl de bodem op het niveau $z = -H(x)$ ligt. Deze geometrische karakteristieken zijn weergegeven in Fig. 1. Zowel breedtevariatie als bodemligging kunnen willekeurig worden gekozen.

Figuur 1 – Geometrie van het getijbekken zoals gebruikt in het geïdealiseerde model, (a): zij aanzicht, (b): bovenaanzicht.



Het onderhavige model kent een aantal belangrijke uitbreidingen ten opzichte van het recente werk van Chernetsky *e.a.*, (2010). Zo kan in principe een willekeurige breedtevariatie van het getijkanaal worden voorgeschreven. Ook kan de bodemruwheid per getijcomponent van de stroming verschillend worden voorgeschreven, hetgeen een realistischere beschrijving van de waterbeweging geeft.

2.1.2 Specifieke aannames

De waterbeweging wordt beschreven door de vergelijking van impulsbalans voor de horizontale snelheid en de vergelijking voor massabehoud. Deze vergelijkingen zijn vanwege hun niet-lineaire karakter in het algemeen niet eenvoudig op te lossen. Door een aantal specifieke veronderstellingen is het echter mogelijk om de vergelijkingen middels een *schalingsprocedure* te vereenvoudigen, waarna het mogelijk wordt om relatief eenvoudig benaderende oplossingen voor de waterbeweging te vinden. Het voordeel van deze vereenvoudigde beschrijving is dat de modelresultaten op een gestructureerde wijze kunnen worden geanalyseerd in termen van fysische mechanismen.

Waterbeweging De eerste vereenvoudiging die wordt toegepast is de aanname dat de amplitude van het verticale M_2 -getij klein is ten opzichte van de typische bodemdiepte. Specifiek wordt dit weergegeven door de conditie

$$\epsilon = \frac{A_{M_2}}{H_0} \ll 1 ,$$

waarbij $H_0 = H(x = 0)$ de breedtegemiddelde diepte ter hoogte van het zeegat voorstelt. Deze aanname is doorgaans gerechtvaardigd in relatief diepe delen van het bekken, maar kan minder nauwkeurig zijn in ondiepe gebieden nabij de landwaartse rand.

Tevens wordt aangenomen dat het extern opgelegde M_4 -getij klein is ten opzichte van de M_2 -forcing in de zin dat de verhouding tussen beide getij-amplitudes ongeveer van dezelfde grootte-orde is als ϵ .

Tot slot wordt er ook van uitgegaan dat de waterbeweging getijgedomineerd is, specifiek wordt aangenomen dat de typische grootte van optredende reststromingen (bijvoorbeeld door rivierafvoer) een orde ϵ kleiner is dan de M_2 -stroomsnelheid. Deze aanname is technisch gezien niet juist nabij de stuw aangezien de getijsnelheid daar nul is maar de rivierafvoer niet. In de praktijk is dit echter geen groot probleem aangezien het model de rivierstroming nabij de stuw wel correct oplost.

Turbulentiemodel De interne wrijving die de stroming ondervindt wordt beschreven door het vereenvoudigde algebraïsche turbulentiemodel van Chernetsky *e.a.*, (2010). Hierbij wordt de verticale viscositeit A_v uitsluitend afhankelijk genomen van de bodemdiepte via

$$A_v(x) = A_{v_0} \left(\frac{H(x)}{H_0} \right)^m , \quad (2)$$

waarbij A_{v_0} constant is. De bodemruwheid wordt beschreven middels de parameter s_f die op een soortgelijke wijze van de bodemdiepte afhangt via

$$s_f = s_{f_0} \left(\frac{H(x)}{H_0} \right)^n , \quad (3)$$

waarbij s_{f_0} wederom constant is. Beide parameters zijn onderling gerelateerd door de bodemrandvoorwaarde voor de horizontale snelheid

$$A_v \frac{\partial u}{\partial z} = s_f u , \quad (4)$$

zodat de bodemschuifspanning lineair afhangt van snelheid aan de bodem. Conditie (4) wordt ook wel de *partiële slip* randvoorwaarde genoemd.

2.1.3 Aanvullende veronderstellingen

Naast de aannames dat het de verticale getij-amplitude klein is ten opzichte van de bodemdiepte en de stroming getij-gedomineerd is zijn er nog een zestal aanvullende veronderstellingen die er feitelijk op neerkomen dat specifieke effecten (vooralsnog) niet worden meegenomen. Deze aanvullende aannames zijn

1. het zoutveld is tijdsafhankelijk en diagnostisch (dwz. voorgeschreven) en vertikaal goed doorgemengd,
2. de invloed van stratificatie op verticale menging wordt niet beschouwd,
3. het effect van platen (droogvallende gebieden) is niet meegenomen,
4. geen krommingseffecten,
5. geen windeffecten,
6. geen golfwerking.

2.1.4 Oplossingsmethode

Door bovengenoemde aannames is het mogelijk om benaderende oplossingen voor de waterbeweging te verkrijgen. In het bijzonder geldt dat de stroomsnelheid en waterstand in eerste benadering door het externe M_2 -getij worden gedreven, terwijl de rest -en M_4 -component van een relatieve grootte ϵ zijn. Dit kan wiskundig worden uitgedrukt door de expansie

$$\begin{aligned}\zeta &= \zeta_0 + \zeta_1 + \dots, \\ u &= u_0 + u_1 + \dots, \\ w &= w_0 + w_1 + \dots,\end{aligned}$$

waarbij grootheden met subindex 0 enkel met de M_2 -periode variëren terwijl grootheden met subindex 1 residuele -en M_4 -bijdragen kennen. Bovengenoemde benaderingen lopen in feite oneindig lang door, en er kunnen dus ook bijdragen met relatieve sterkte ϵ^2 en ϵ^3 worden gevonden. Deze worden niet in het onderhavige model opgelost, maar zullen wel worden bestudeerd in het kader van WL-project 15_039 "Uitbreiding geïdealiseerd procesmodel ondiep water" om het effect van ondiepere gebieden nauwkeuriger mee te kunnen nemen.

2.2 Gebruikte gebiedsgegevens

2.2.1 Geometrie

Het model is toegepast op het gehele Scheldebekken van Vlissingen tot Gentbrugge, het betreft hier een gebied met een lengte van 160 km. De gebruikte topo-bathymetrische data zijn afkomstig van het 2DH-model dat door Coen *e.a.*, (2016) is beschreven. Het betreft hier de meeste recente bodemgegevens die geheel gebiedsdekkend zijn. Op basis hiervan is de breedtevariatie $B(x)$ op het niveau van gemiddeld halftij bepaald alsmede de bijbehorende breedtegemiddelde bodemligging $H(x)$. Deze vertonen echter nog een zeer grillig langsverloop die niet in een geïdealiseerd model kan worden meegenomen (Chernetsky *e.a.*, 2010). Er wordt daarom gebruikt gemaakt van gladdere fits. Voor de breedtefit $p_B(x)$ is gekozen voor een exponent van een rationale functie

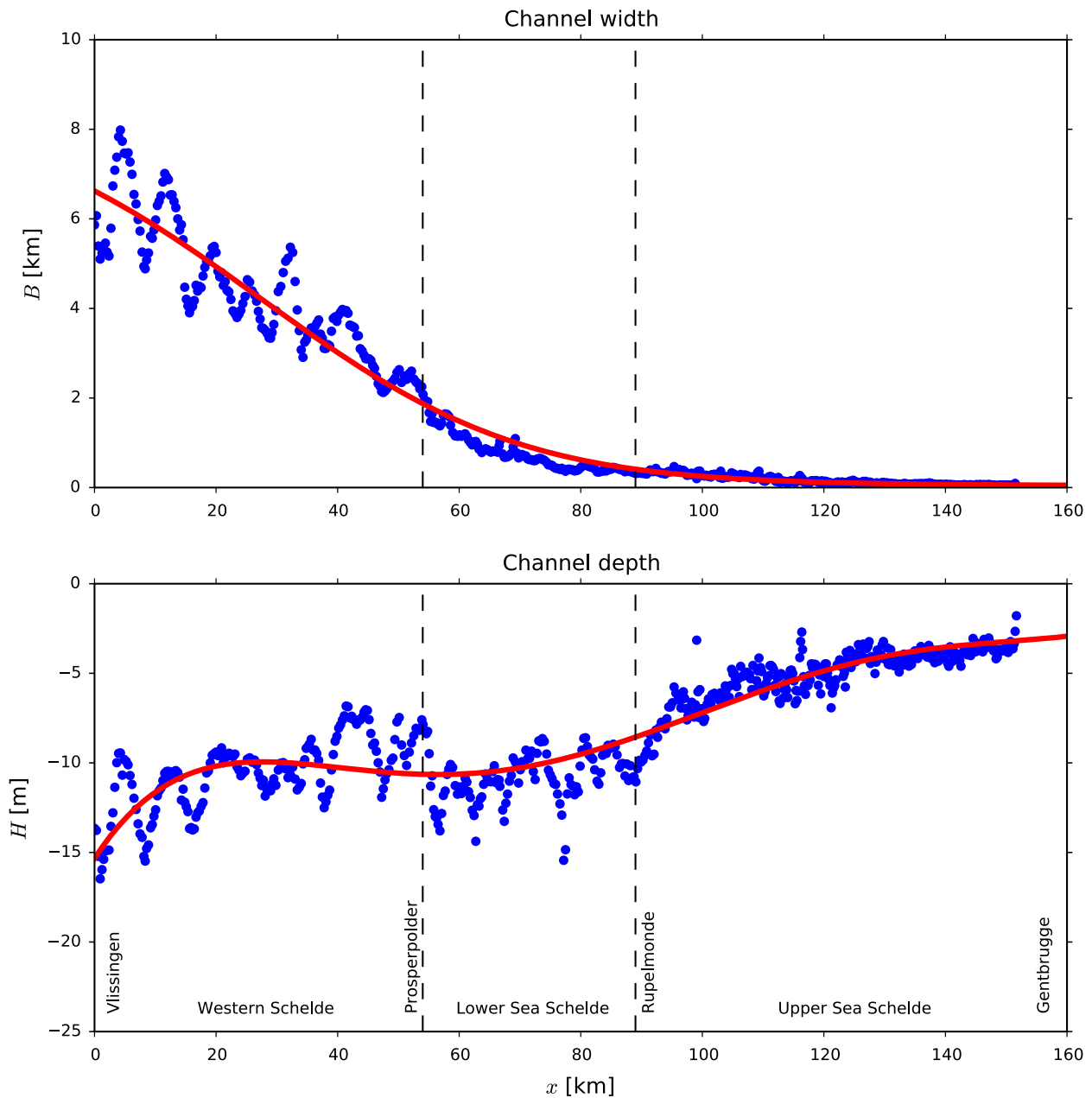
$$p_B(x) = \exp\left(\frac{k_1x + k_2}{k_3x^2 + k_4x + k_5}\right), \quad (5)$$

waarbij $p_B(x)$ is uitgedrukt in km en x in m. De functionele vorm (5) is gekozen om algemenere breedtevariatiën toe te laten dan het vaak gebruikte louter exponentiële verloop. Deze is voor de Schelde namelijk minder accuraat aangezien de exponentiële convergentielengte voor het Scheldebekken niet constant is (Kuijper en Van Rijn, 2011). Voor de fit van de bodemdiepte $p_H(x)$ is een vijfde orde polynoom is gebruikt:

$$p_H(x) = d_1x^5 + d_2x^4 + d_3x^3 + d_4x^2 + d_5x + d_6. \quad (6)$$

Hierbij zijn $p_H(x)$ en x uitgedrukt in meter. In Fig. 2 zijn de gladde benaderingen $p_B(x)$ en $p_H(x)$ grafisch weergegeven.

Figuur 2 – Kanaalbreedte (boven) en breedtegemiddelde bodemligging (onder) als functie van positie in het Schelde estuarium.



De blauwe punten zijn de langsvariatie van het model van Coen *e.a.*, 2016, de rode lijnen de beste fits volgens Vgl. (5) resp. (6).

De waarden voor de parameters in Vgl. (5) en (6) voor de geometrie zoals weergegeven in Fig. 2 staan in Tabel 1 vermeld.

2.2.2 Waterstanden

Op de open zeerand wordt randvoorwaarde (1) opgelegd met $A_{M_2} = 1.77$ m, $A_{M_4} = 0.14$ m en $\varphi = -1.3^\circ$ de M_2 -en M_4 astronomische getijcomponenten te Vlissingen. De amplitude en fase van deze componenten

Tabel 1 – Coëfficiënten behorende bij de beste fits voor Vgl. (5) en (6) zoals getoond in Fig. 2.

Parameter	Eenheid	Waarde
k_1	m^{-1}	-2.742×10^{-5}
k_2	—	1.8973
k_3	m^{-2}	4.9788×10^{-11}
k_4	m^{-1}	-9.213×10^{-6}
k_5	—	1
d_1	m^{-4}	-2.9013×10^{-24}
d_2	m^{-3}	1.4030×10^{-18}
d_3	m^{-2}	-2.4218×10^{-13}
d_4	m^{-1}	1.7490×10^{-8}
d_5	—	-5.2141×10^{-4}
d_6	m	15.332

zijn in Tabel 2 zijn voor diverse lokaties langs de Schelde weergegeven. Deze harmonische data zijn bepaald uit tijdreeksen van het jaar 2009 (Maximova *e.a.*, 2013) met behulp van het programma T_TIDE (Pawlowicz *e.a.*, 2002). De gegevens in Tabel 2 zijn gebruikt voor de calibratie van het model.

De amplitude van de M_2 -waterstand in het meest stroomopwaartse deel van het Scheldebekken (stations

Tabel 2 – Amplitude en fase van het M_2 -en M_4 getij voor diverse waterstandsstations in het Schelde estuarium.

Locatie	Afstand tot Vlissingen (km)	M_2 amplitude (m)	M_2 fase (°)	M_4 amplitude (m)	M_4 fase (°)
Vlissingen	0	1.77	0	0.14	-1.3
Terneuzen	18.5	1.98	10.4	0.12	12.3
Hansweert	33.8	2.03	20.5	0.11	39.4
Bath	49.8	2.18	31.1	0.11	58.7
Prosperpolder	54	2.19	33.1	0.12	62.2
Liefkenshoek	61.1	2.26	35.6	0.12	65.1
Antwerpen	75.6	2.31	44.3	0.13	74.6
Temse	97.3	2.28	62.7	0.16	88.2
St. Amands	106.8	2.22	73.4	0.24	101.7
Dendermonde	119.8	1.69	93.3	0.25	128.7
Schoonaarde	130.6	1.31	116.4	0.24	164.3
Wetteren	142.7	1.09	143.4	0.21	212.6
Melle	148.8	1.02	157.4	0.22	242.9

De fasen zijn gemeten ten opzichte van hoogwater van het M_2 -getij te Vlissingen.

Wetteren en Melle) kent overigens een seizoensafhankelijke variatie van 5 à 10 cm vanwege de invloed van rivierafvoer (Schramkowski *e.a.*, 2017).

2.2.3 Snelheden

Er is gebruik gemaakt van dezelfde snelheidsgegevens als Nnafie *e.a.*, (2016). Dit betreft metingen die middels een gesleepte ADCP zijn verricht langsheen een elftal raaien. De locatie en tijdstippen van deze metingen zijn in Tabel 3 opgelijst. Uit de gemeten drie-dimensionale snelheidsvelden zijn door middeling over een sectie de cross-sectioneel gemiddelde horizontale snelheidscomponenten bepaald. De component langsheen het bekken is tot slot berekend door deze snelheidsvector te projecteren langs de curve die de kilometrering van de Schelde weergeeft.

Van de aldus verkregen longitudinale snelheid zijn de harmonische componenten M_0 (restsignaal), M_2 en M_4 bepaald door gebruik te maken van een kleinste kwadratenbenadering. Deze benadering en de vergelijking met de gemeten langssnelheden worden in Appendix A2 nader besproken.

Tabel 3 – Locatie en tijdstip van de gebruikte snelheidsmetingen in de Schelde.

Locatie	Afstand tot Vlissingen (km)	Start meetcampagne	Einde meetcampagne
Terneuzen	17	06-Jul-2011 05:38:19	06-Jul-2011 18:10:22
Gat van Ossenisse	28	09-Mei-2012 05:01:36	09-Mei-2012 17:47:13
Schaar van de Waarde	40	24-Apr-2013 05:39:17	24-Apr-2013 18:26:28
Zuidergat	40	25-Apr-2013 06:22:56	25-Apr-2013 19:09:42
Liefkenshoek	63	25-Jun-2013 06:29:45	25-Jun-2013 19:16:03
Oosterweel	74	27-Jun-2013 06:26:47	27-Jun-2013 19:03:47
Kruikeke	85	30-Mei-2013 06:11:14	30-Mei-2013 19:06:25
Driegoten	102	12-Jun-2013 06:54:02	12-Jun-2013 19:14:04
Dendermonde	120	17-Apr-2014 07:50:51	17-Apr-2014 19:50:00
Schoonaarde	132	27-Mei-2013 08:18:12	27-Mei-2013 20:05:01
Schellebelle	140	15-Apr-2014 07:58:24	15-Apr-2014 19:10:29

2.2.4 Zoutverdeling en rivierafvoer

Er zijn getijgemiddelde zoutverdelingen gebruikt die zijn gebaseerd op meetcampagnes uit februari en oktober 2009 (zie Schramkowski *e.a.*, 2017). Deze situaties komen overeen met rivierafvoeren $Q = 110 \text{ m}^3 \text{ s}^{-1}$ respectievelijk $Q = 15 \text{ m}^3 \text{ s}^{-1}$. De situatie uit februari 2009 is typisch voor wintercondities, terwijl oktober 2009 representatief is voor zomercondities.

De metingen in februari betroffen een meting tijdens snelheidskentering na hoogwater op 4 februari en een kentering na laagwater op 10 februari. Uitgaande van deze conductiviteitsmetingen is het zoutgehalte bepaald met behulp van de UNESCO-formulering (Fofonoff en Millard, 1983), waarbij een watertemperatuur van 7°C is aangenomen op basis van de gegevens die in Fettweis *e.a.*, (1997) staan vermeld. Vervolgens is de getijgemiddelde saliniteit per station verkregen door de metingen van 4 en 10 februari te middelen.

Voor de metingen van oktober geldt analoog dat er op 5 oktober conductiviteit is bepaald tijdens een snelheidskentering na laagwater en op 13 oktober tijdens een kentering na hoogwater. De saliniteit is op dezelfde wijze bepaald als voor de gegevens uit februari, waarbij nu een watertemperatuur van 19°C is aangenomen (Fettweis *e.a.*, 1997).

De gemeten getijgemiddelde zoutverdeling $s(x)$ wordt in het model voorgeschreven middels een fit die ook in de literatuur wordt gebruikt (zie Warner *e.a.*, 2005; Talke *e.a.*, 2009), te weten

$$s(x) = \frac{\hat{s}}{2} \left[1 - \tanh \left(\frac{x - x_c}{x_L} \right) \right]. \quad (7)$$

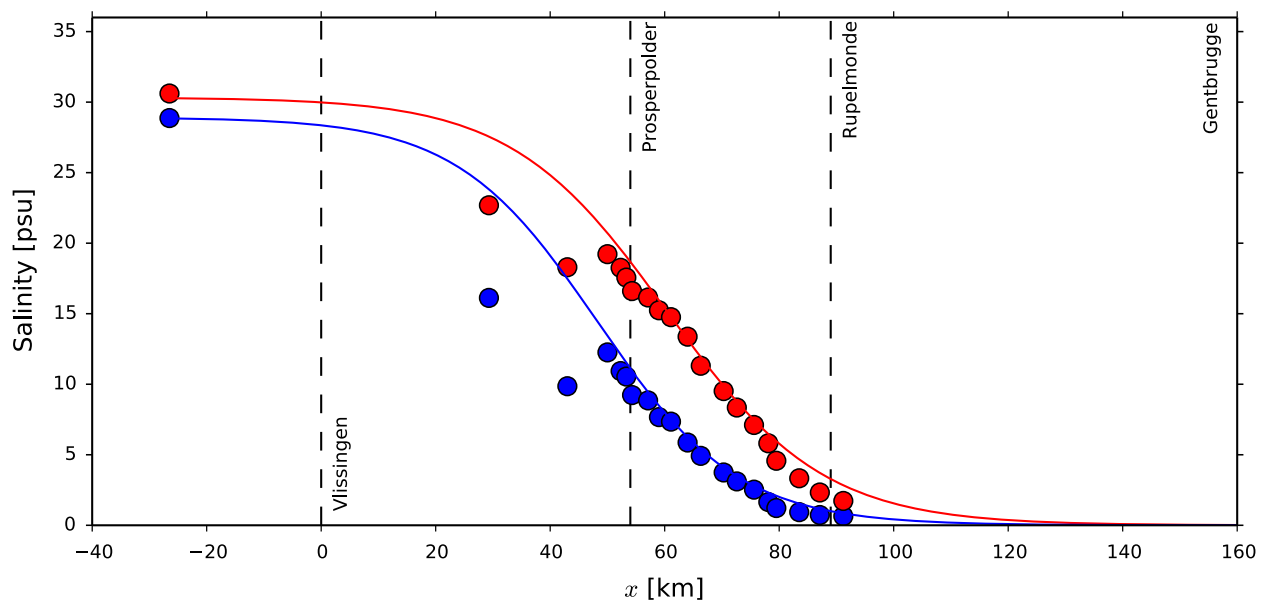
De parameters $\hat{s}$, x_c en x_L zijn voor winter -en zomercondities afzonderlijk bepaald, zie Tabel 4. De betekenis van deze grootheden is als volgt: $\hat{s}$ is de zoutconcentratie op open zee, x_c is de locatie van de maximale zoutgradiënt terwijl x_L een maat is voor de afmeting van het overgangsgebied tussen zoet rivierwater en zout zeewater. Gebruik van fit (7) veronderstelt dat de waterkolom gedurende een getijcyclus goed is doorgemengd, dat wil zeggen dat verticale variaties in de zoutverdeling klein zijn ten opzichte van de dieptegemiddelde waarde.

Tabel 4 – Parameterwaarden behorende bij voorschrift (7).

Parameter	Wintercondities	Zomercondities
$\hat{s}$ (psu)	28.9131	30.3296
x_c (km)	48.2068	60.4140
x_L (km)	24.5097	27.1020

De gemeten zoutwaarden en het bijbehorende verloop volgens Vgl. (7) zijn in Fig. 3 voor zowel zomer -als wintercondities weergegeven.

Figuur 3 – Longitudinale verdeling van de breedte -en getijgemiddelde zoutverdeling in het Scheldebekken.



De punten geven de gemeten waarden weer voor zomer (blauw) en winter (rood). De blauwe en rode lijnen zijn de bijbehorende beste fits volgens Vgl. (7) op basis van Tabel 4. Het gebied met negatieve x -waarden is geen onderdeel van het model domein.

2.3 Modelcalibratie

Het model is gecalibreerd door optimale waarden voor de turbulentieparameters A_{v_0} en s_{f_0} te bepalen zodanig dat de waargenomen M_2 en M_4 -waterstanden zo goed mogelijk worden beschreven. Hierbij is de dieptevariatie in Vgl. (2) en (3) niet meegenomen (m en n zijn dus beiden nul gesteld) aangezien dit geen meerwaarde bleek te geven bij de afregeling.

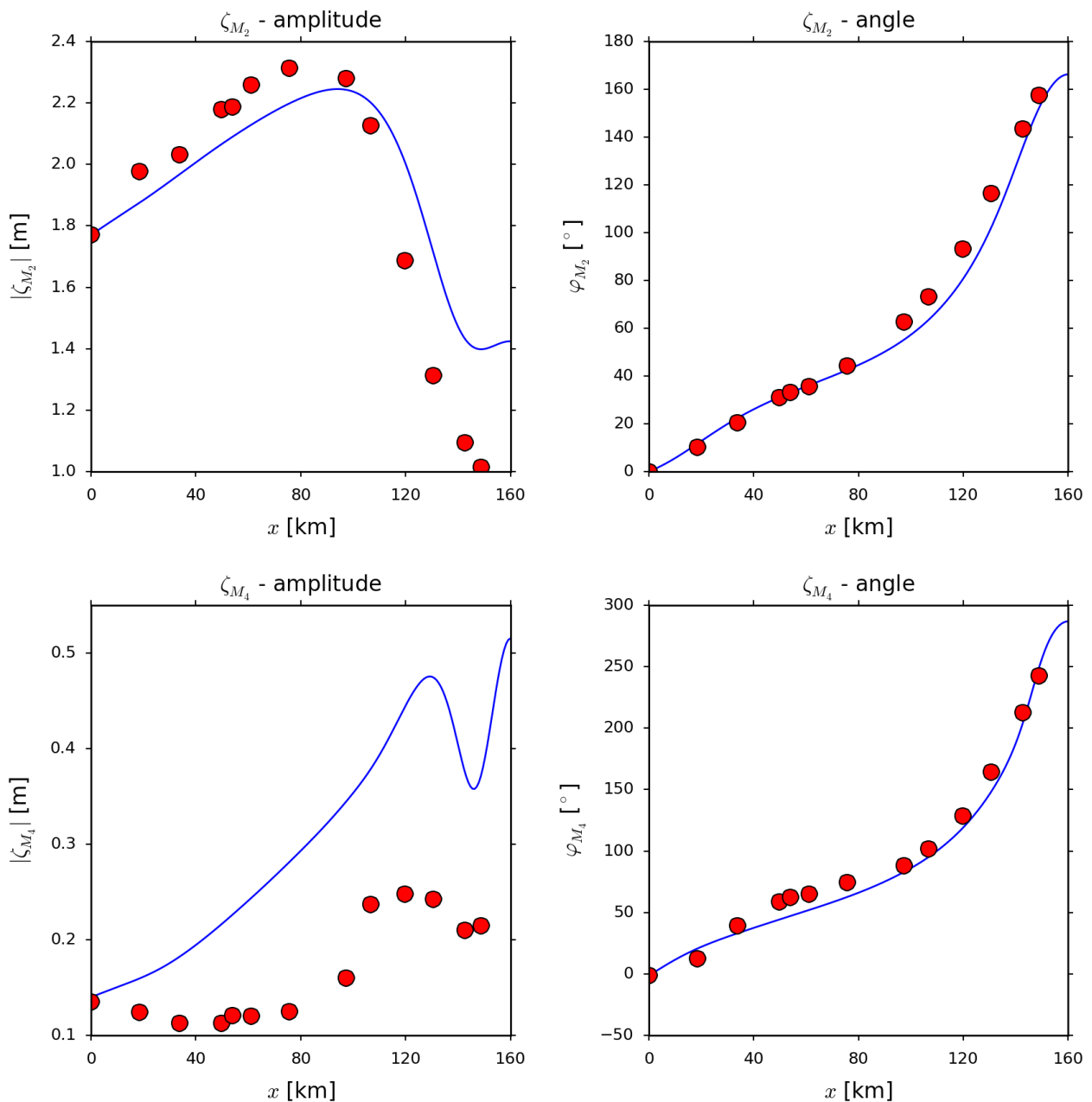
Daarnaast is eerder vastgesteld (Schramkowski *e.a.*, 2017) dat het M_4 -getij door het model waarschijnlijk overschat wordt vanwege de gebruikte bodemrandvoorwaarde (4). Deze overschatting komt door de aanname dat

s_f voor alle getijcomponenten gelijk is. Echter, zoals in Hoofdstuk 5 wordt besproken is de bodemruwheid zoals geformuleerd in Vgl. (4) het gevolg van een linearisatieprocedure en derhalve afhankelijk van de beschouwde getijcomponent (Ianniello, 1977). Om deze reden is de bodemruwheid voor M_0 en M_4 tweemaal groter genomen dan voor M_2 , dus

$$s_{f_{0,M_0}} = s_{f_{0,M_4}} = 2s_{f_{0,M_2}}.$$

Het blijkt dat de meest optimale waarden voor de turbulentieparameters $A_{v_0} = 0.061 \text{ m}^2 \text{ s}^{-1}$ en $s_{f_{0,M_2}} = 0.003 \text{ m s}^{-1}$ zijn. De bijbehorende verdelingen van amplitude en fase langsheen het Scheldebekken zijn in Fig. 4 weergegeven.

Figuur 4 – Amplitude (links) en fase (rechts) van het gecalibreerde M_2 (boven) -en M_4 -getij (onder).



Rode punten geven metingen weer en blauwe lijnen corresponderen met modelresultaten.

De door het model berekende fase van het verticale M_2 -en M_4 -getij komt kwantitatief goed overeen met de metingen. Voor de amplitude van het M_2 -getij geldt het kwalitatieve verloop langs het bekken (eerst ver-

sterking, meer stroomopwaarts sterke damping) goed wordt gereproduceerd. De gemodelleerde piek in M_2 -amplitude is wat landwaarts verschoven ten opzichte van de metingen. Tot slot is de M_4 -amplitude van het gecalibreerde model van dezelfde grootte-orde als de metingen, hoewel er in de Westerschelde en Beneden Zeeschelde nog steeds sprake is van een aanmerkelijke overschatting door het model. Tot slot wijken zowel de M_2 - als M_4 -amplitude ter hoogte van de landwaartse rand (stuw) behoorlijk af van de waarnemingen.

2.4 Resultaten

2.4.1 Asymmetrie van het verticale getij

Het verloop van de waterstand is in het algemeen asymmetrisch, dat wil zeggen dat bijvoorbeeld de tijdsduur van stijging t_{fall} (van hoog -naar laagwater) en daling t_{rise} (van laag -naar hoogwater) ongelijk zijn. Deze soort getij-asymmetrie kan worden gekwantificeerd door de parameter δ_{asym} welke is gedefinieerd als

$$\delta_{\text{asym}} = \frac{t_{\text{fall}}}{t_{\text{rise}}} . \quad (8)$$

Voor $\delta_{\text{asym}} > 1$ duurt stijging korter dan daling, terwijl bij $\delta_{\text{asym}} < 1$ de daling korter duurt (Friedrichs, 2010). Het verloop van de M_2 - en M_4 -waterstandscomponenten langsheen het Schelde estuarium kan worden weergegeven als

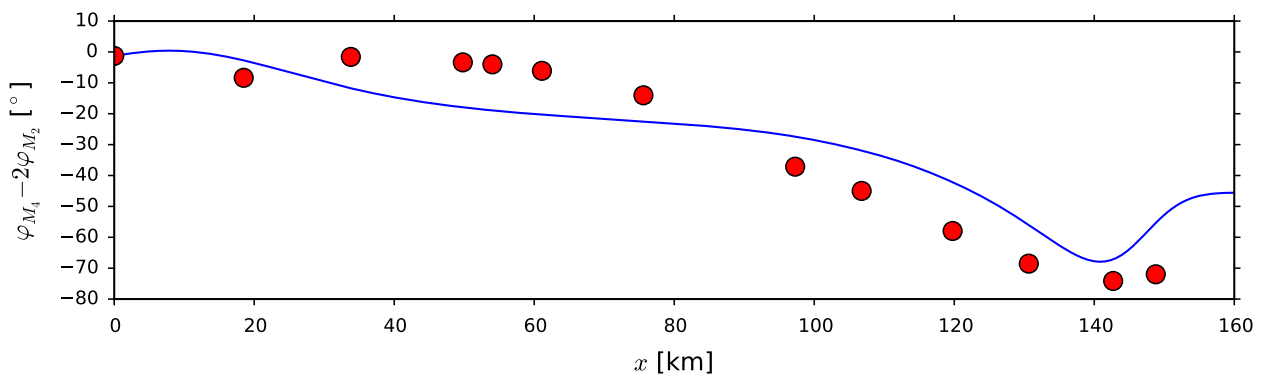
$$\zeta(x, t) = A_{M_2}(x) \cos(\sigma t - \varphi_{M_2}(x)) + A_{M_4} \cos(2\sigma t - \varphi_{M_4}(x)) ,$$

en het faseverschil $\Delta\varphi$ tussen beide getijcomponenten is gegeven door $\Delta\varphi = \varphi_{M_4} - 2\varphi_{M_2}$. Voor het onderhavige model (waarin het M_4 -getij relatief zwak is ten opzichte van M_2) kan δ_{asym} aan $\Delta\varphi$ worden gerelateerd via

$$\delta_{\text{asym}} = 1 - \frac{8}{\pi} \frac{A_{M_4}(x)}{A_{M_2}(x)} \sin(\Delta\varphi) .$$

In Fig. 5 is het faseverschil tussen M_2 - en M_4 -getij langs het Scheldebekken weergegeven. Hoewel er tussen modelresultaat en metingen afwijkingen tot ca 20° optreden, is het faseverschil als zodanig tussen –80 en nul graden. Dit betekent dat $\delta_{\text{asym}} > 1$ zodat de stijging van het getij korter duurt dan de daling. Dit resultaat komt kwalitatief goed overeen met de analyse van gemeten hoog -en laagwaters die ook een getij impliceren waarbij stijging korter duurt dan daling (Vandenbruwaene *e.a.*, 2016a). Het geïdealiseerde model geeft dus een redelijk goede beschrijving van de asymmetrie van het verticale getij.

Figuur 5 – Faseverschil tussen het verticale M_2 - en M_4 -getij als functie van de locatie langs het Schelde estuarium.

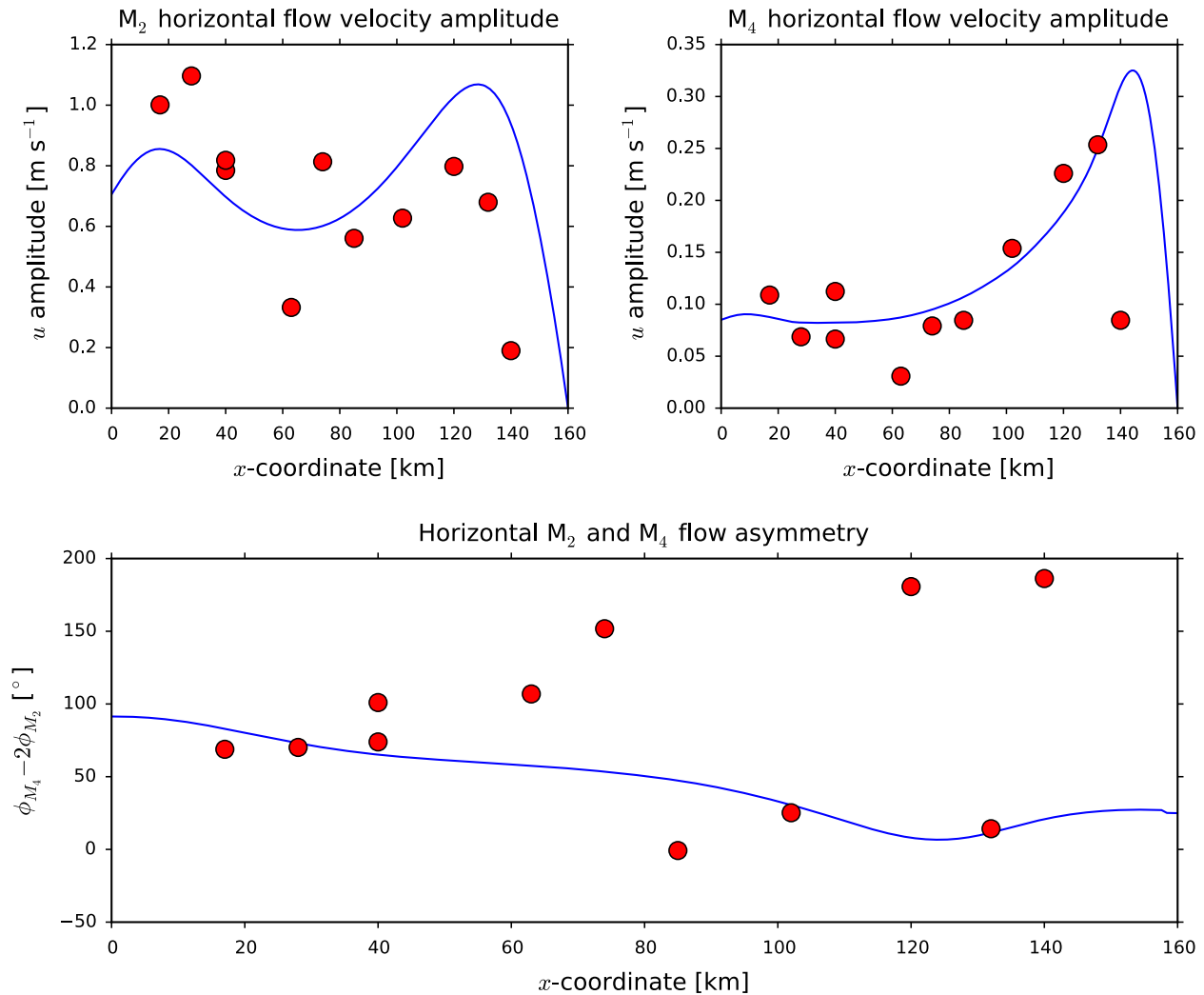


Rode punten geven metingen weer, de blauwe lijn het modelresultaat.

2.4.2 Getijstroming

Amplitudes van de stromingscomponenten In de bovenste twee panelen van Fig. 6 is een vergelijking weergegeven tussen de waargenomen amplitudes van de horizontale M_2 -en M_4 -stroming (zie Appendix A2) enerzijds en de modelresultaten anderzijds. Hoewel de amplitudes qua orde van grootte goed overeenkomen is er geen nauwkeurige trendmatige overeenkomst te zien. De M_4 -snelheid in het achterste deel van het bekken wordt door het model overschat.

Figuur 6 – Vergelijking tussen berekende en waargenomen getijstroming.



Boven: amplitude van de dieptegemiddelde M_2 -snelheid (links) en de dieptegemiddelde M_4 -snelheid (rechts) langsheen het Scheldebekken. De onderste paneel geeft de langsvariatie van het faseverschil $\Delta\phi$ tussen de M_2 -en M_4 stromingscomponent. Rode bollen verwijzen naar waarnemingen terwijl blauwe lijnen modelresultaten zijn.

Asymmetrie van het horizontale getij Net als bij het verticale getij (waterstanden) kan de dieptegemiddelde getijsnelheid $\bar{u}$ langs het bekken worden benaderd door

$$\bar{u}(x, t) = \bar{u}_{M_2}(x) \cos(\sigma t - \phi_{M_2}(x)) + \bar{u}_{M_4}(x) \cos(2\sigma t - \phi_{M_4}(x)). \quad (9)$$

Deze stroming is doorgaans asymmetrisch, dat wil zeggen dat de typische eb- en vloed-snelheid in het algemeen verschillend zullen zijn. Dit heeft implicaties voor met name netto sedimenttransport, en daarmee ook voor

zaken als globale slibbalans en de dynamica van estuariene troebelheidsmaxima. Ook bij de omslaghypothese die voor het ontstaan van hypertroebelheid is voorgesteld speelt asymmetrie van getijsnelheid een belangrijke rol (Winterwerp en Wang, 2013).

Het effect van horizontale getij-assymetrie op sedimenttransport kan worden geïllustreerd door aan te nemen dat de instantane sedimentflux q_{sed} op cubische wijze afhangt van de dieptegemiddelde snelheid volgens $q_{\text{sed}} = p\bar{u}^3$. De netto (dus getij-gemiddelde) sedimentflux $\langle q_{\text{sed}} \rangle$ is dan met snelheidsveld (9) gegeven door de uitdrukking

$$\langle q_{\text{sed}} \rangle = \frac{3}{4} p \bar{u}_{M_2}^2 \bar{u}_{M_4} \cos(\Delta\phi), \quad (10)$$

waarbij $\Delta\phi = \phi_{M_4} - 2\phi_{M_2}$ het faseverschil is tussen de horizontale M_2 -en M_4 -getijsnelheid. Als $\langle q_{\text{sed}} \rangle$ positief is, dan is de netto flux landwaarts gericht en spreekt men van vloeddominatie. Dit treedt op als $\Delta\phi$ tussen -90° en 90° ligt. Voor negatieve waarden van $\langle q_{\text{sed}} \rangle$ is de netto sedimentflux zeewaarts gericht en is er sprake van ebdominatie. Alleen voor $\Delta = \pm 90^\circ$ is er een symmetrisch horizontaal getij zonder netto sedimentflux.

Het onderste paneel van Fig. 6 laat een vergelijking zien tussen het berekende verloop van $\Delta\phi$ langs het estuarium en de waarden bepaald uit de snelheidsmetingen (zie Appendix A2). De overeenkomst is redelijk goed voor de Westerschelde ($x < 60\text{km}$), waar het getij redelijk symmetrisch is ($\Delta\phi \approx 90^\circ$). Meer landwaarts geeft het model aan dat er sprake is van vloeddominatie terwijl de metingen hier een grote spreiding laten zien.

De modelresultaten komen voor wat betreft het kwalitatieve verloop van de getij-asymmetrie (vloeddominatie in de Zeeschelde) goed overeen met de resultaten van Winterwerp *e.a.*, (2015) die gebaseerd zijn op numerieke berekeningen met Delft3D.

Dominante bijdragen aan de M_4 -snelheid In Fig. 7 is het longitudinale en vertikale verloop van de amplitude en fase van de M_4 -snelheid weergegeven en zijn ook de individuele bijdrage van verschillende mechanismen aangegeven. Hieruit blijkt dat er een M_4 snelheidsmaximum in de Westerschelde is. De M_4 -snelheid wordt hier vooral bepaald door het externe getij te Vlissingen. In de Boven Zeeschelde treedt een tweede lokale maximum van de M_4 snelheid op. Dit maximum wordt grotendeels veroorzaakt door de Stokes-stroming (correlatie tussen M_2 -waterstand en M_2 -snelheid) en door de schuifspanning aan het wateroppervlak.

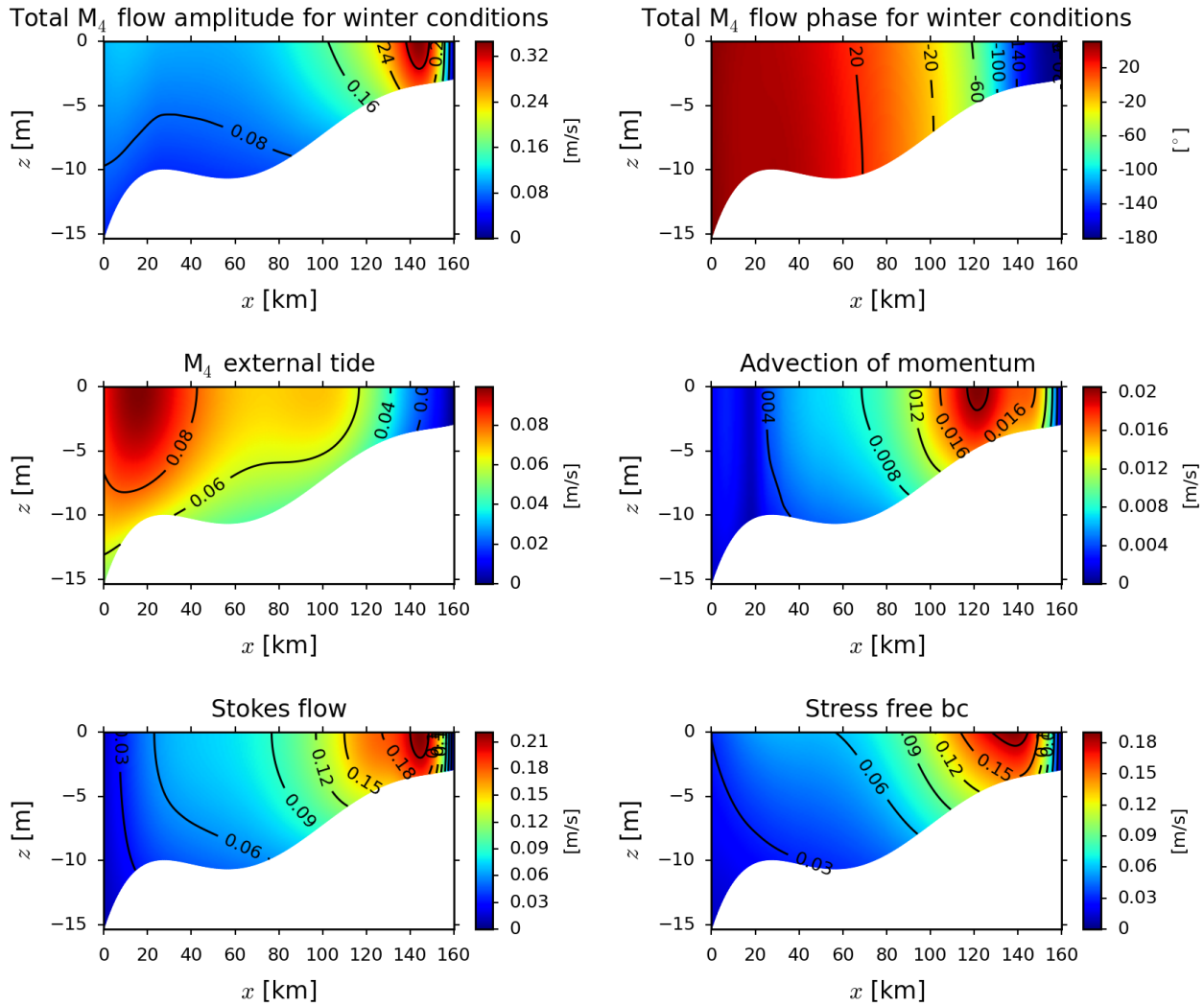
2.4.3 Reststroming

Stroomcomponenten Het geïdealiseerde model kan ook de residuele (of rest) stroming uitrekenen, dit is het tijdsafhankelijke deel van de waterbeweging. De reststroming kent diverse componenten, waarvan we er hier drie zullen bespreken.

De eerste is de reststroming ten gevolge van de rivierafvoer. Dit is de totale stroming als er geen getij of zout-verdeling is. De tweede bijdrage is de gravitationele circulatie, welke het gevolg is van de variatie van het zoutgehalte langsheen het getijbekken. Deze stroming is landwaarts gericht nabij de bodem en zeewaarts nabij het wateroppervlak, terwijl de vertikaal gemiddelde circulatiesnelheid nul is. De gravitationele circulatie transporteert dus netto geen water.

De derde bijdrage is de Stokes terugkeer-stroming. Deze treedt op als de getijgolf deels een lopend karakter heeft. Er wordt dan tijdens hoogwater meer water landwaarts getransporteerd dan tijdens laagwater zee- waarts. Het gevolg is een netto landwaarts transport van water dat aanleiding zou geven tot een voortdu- rende accumulatie van water nabij de stuw. Dit treedt niet op omdat er een zeewaarts gerichte reststroming wordt opgezet die het landwaartse transport balanceert. Deze zeewaartse stroming wordt de Stokes terugkeer- stroming genoemd.

Wintercondities In Fig. 8 is de totale reststroming als functie van diepte en afstand langs de Schelde weer- gegeven voor de in Sect. 2.2.4 besproken wintercondities. Ook zijn hierbij vijf individuele bijdragen aange-

Figuur 7 – Amplitude (linksboven) en fase (rechtsboven) van de M_4 -snelheid als functie van positie in het Scheldebekken.


De andere vier panelen geven de individuele bijdragen aan de M_4 -snelheid weer als gevolg van het externe M_4 -getij (midden links), de versnellings termen (midden rechts), de Stokesstrooming (linksonder) en schuifspanning aan het wateroppervlak (rechtsonder).

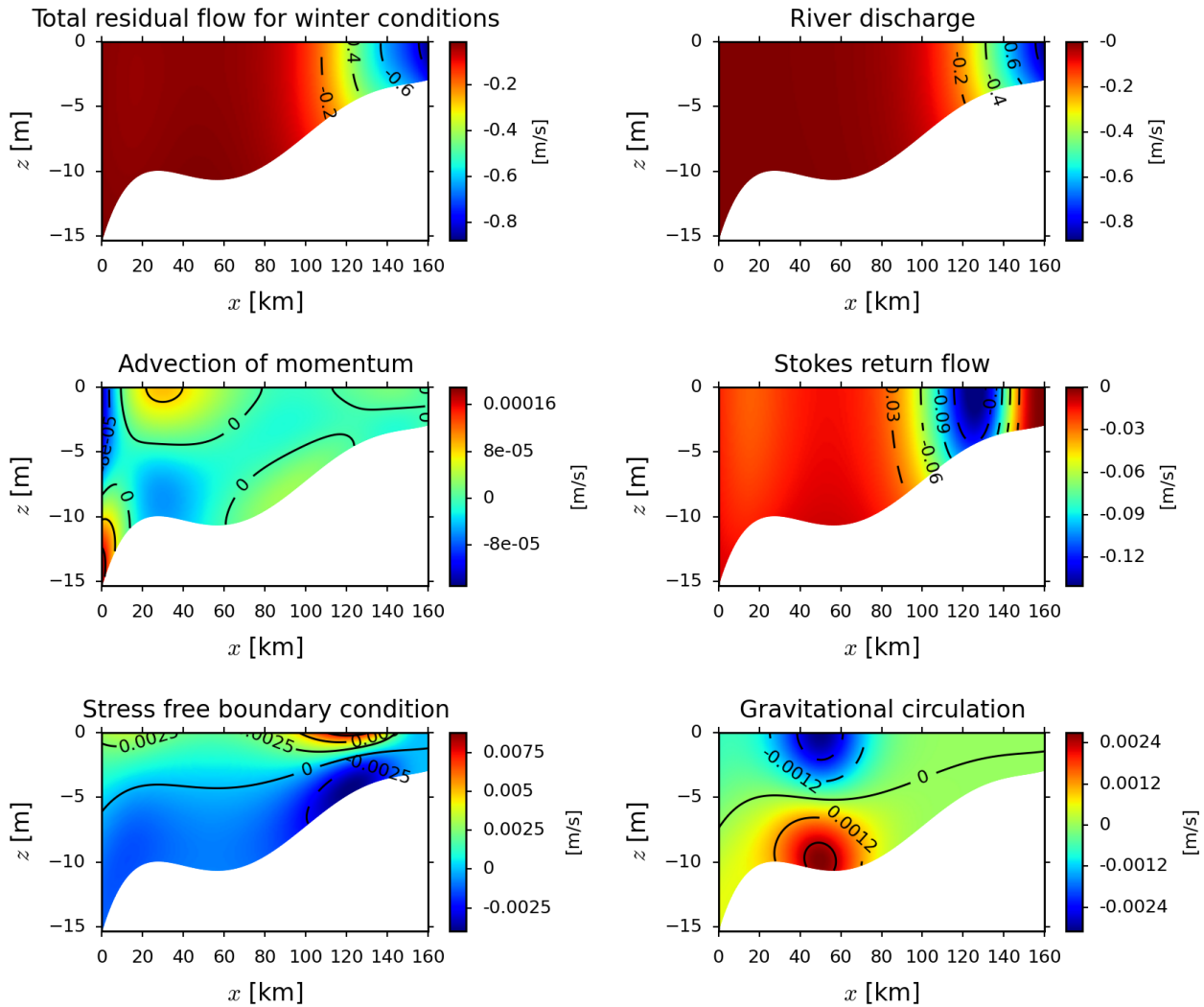
geven. Het blijkt dat in dit geval de totale reststrooming wordt gedomineerd door de rivierbijdrage (maximale stroomsnelheid $\sim -0.9 \text{ m s}^{-1}$). De bijdrage die vervolgens belangrijk is (Stokes terugkeerstroming) kent een maximale sterkte van slechts -0.15 m s^{-1} .

Zomercondities De bijdragen aan de reststrooming bij zomercondities (zie Sect. 2.2.4) zijn weergegeven in Fig. 9.

De totale reststrooming is nu aanzienlijk lager (maximaal ca. $\sim -0.19 \text{ m s}^{-1}$), hetgeen een directe weergave is van de rivierafvoer die ongeveer een orde van grootte lager is (maximum snelheid $\sim -0.12 \text{ m s}^{-1}$). De dominante bijdrage aan de totale strooming is nu de Stokes terugkeerstroming, die ongeveer een factor twee sterker is dan de rivierstroom (zie Fig. 8).

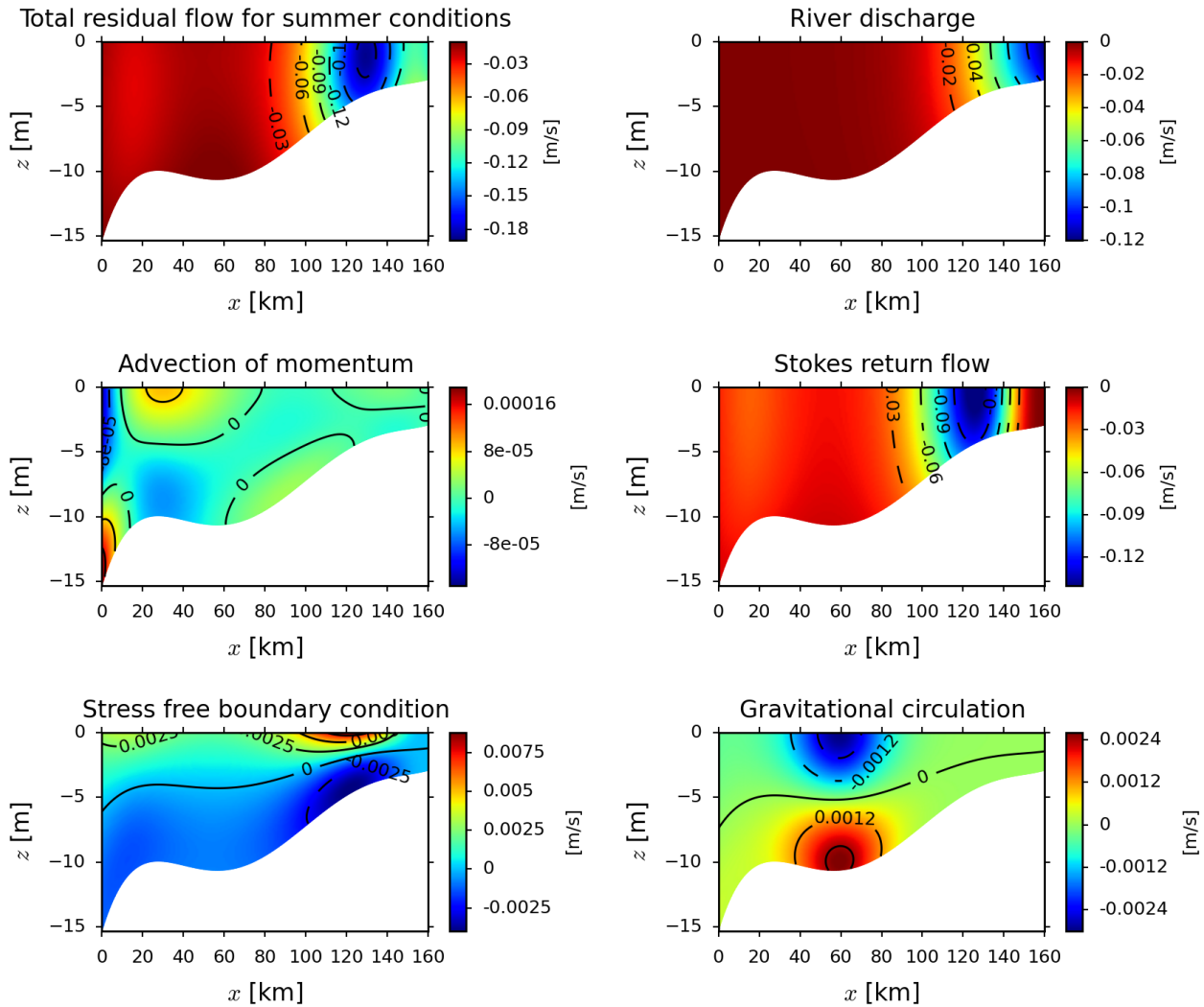
Gravitationele circulatie De gravitationele circulatie geeft zowel tijdens winter -als zomercondities een zeer geringe bijdrage aan de residuele strooming: maximale stroomsnelheden zijn in beide gevallen hooguit 2.5 mm s^{-1} . Enkel de lokatie van de maximale circulatiesnelheid verandert: bij hoge afvoer is zij meer zeewaarts als bij lage

Figuur 8 – Berekende restcomponent van de langgerichte stroming u als functie van locaties x en hoogte z in de Schelde, gebaseerd op wintercondities.



Het paneel linksboven geeft de totale reststroming. De panelen rechts zijn (van boven naar beneden) de rivierbijdrage, Stokes terugkeerstroming en gravitationele circulatie.

afvoer. Dit is in overeenstemming met het feit dat de circulatiesnelheid schaalt als $ds(x)/dx$, en de locatie van deze maximale zoutgradiënt is meer zeewaarts bij hogere debieten (zie Sect. 2.2.4).

Figuur 9 – Berekende reststroming als functie van locaties x en hoogte z voor zomercondities.

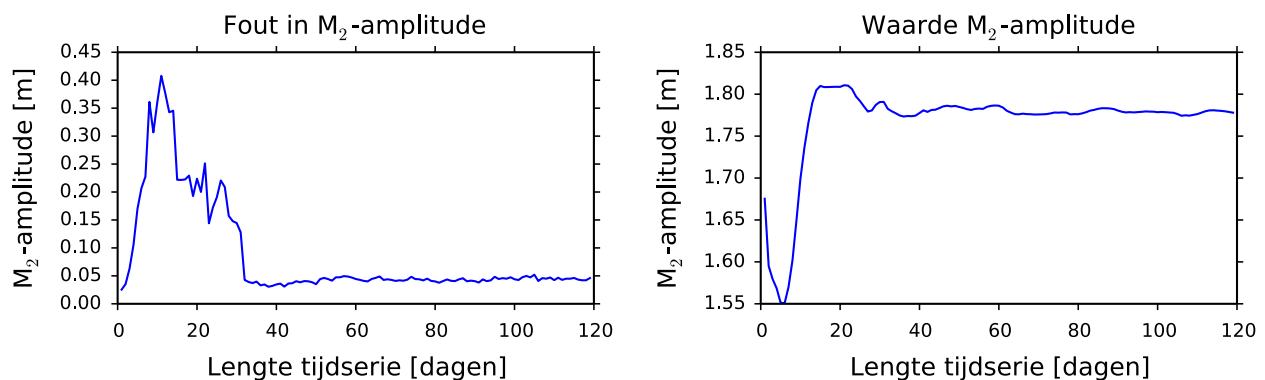
Van boven naar beneden zijn achtereenvolgens de totale reststroming, de rivierbijdrage en gravitationele circulatie getoond.

2.5 Kwaliteit van harmonische analyse van de snelheidsdata

De resultaten in Sect. 2.4 laten zien dat er grote verschillen kunnen optreden tussen de M_2 -en M_4 -getijcomponenten van de snelheid zoals bepaald uit waarnemingen enerzijds en model anderzijds. Waar de modelresultaten een duidelijke trend laten zien, zowel in amplitude als qua asymmetrie (zie Fig. 6), vertonen de metingen een grote spreiding. In de Zeeschelde is op basis van de gebruikte metingen niet te zeggen of er sprake is van eb -danwel vloeddominatie. Deze spreiding is wellicht het gevolg van een gebruik van te korte tijdseries. Meer specifiek lijkt een tijdserie van 13 uur bij lange na niet voldoende voor een nauwkeurige harmonische analyse. Deze uitspraak kan worden gerechtvaardigd door te kijken naar harmonische analyse van waterstanden. Hiervoor is gebruik gemaakt van een tijdserie van gemeten waterstanden voor Vlissingen gedurende de eerste 120 dagen van 2001. Deze meetreeks is gegeven met een interval van 10 minuten. In Fig. 10 zijn de resultaten weergegeven voor de M_2 -amplitude zoals berekend met de kleinste kwadratenprogramma T_TIDE van Pawlowicz *e.a.*, (2002). Dit programma geeft ook een schatting van de fout waarmee een getijcomponent is bepaald. Uit de linkerpaneel van Fig. 10 kan worden afgeleid dat deze fout voor meetseries korter dan ca. dertig dagen vrij groot is (tot ca. 40 cm) om daarna plots af te nemen tot ongeveer 5 cm. Deze fout verandert vervolgens niet meer. In de rechterplaat van Fig. 10 is te zien dat de M_2 -getijamplitude zelf ook pas

bij tijdseries met een lengte van tenminste 30 dagen convergeert naar een min of meer constante waarde. Deze analyse impliceert dat de harmonische componenten van het verticale getij pas nauwkeurig bepaald kunnen worden bij een voldoende lange tijdserie van tenminste een maand. Omdat de methode voor harmonische analyse van de snelheid dezelfde is als voor waterstand roept dit de vraag op of bij tijdseries van ca 13 uur wel nauwkeurig getijcomponenten van de snelheid bepaald kunnen worden. Dit zou betekenen dat de grote spreiding in de waargenomen horizontale getij-asymmetrie en het grillige verloop van de amplitudes van M_2 -en M_4 -snelheden (Fig. 6) veroorzaakt worden door een intrinsiek grote fout in de harmonische analyse. De oplossing ligt klaarblijkelijk in het gebruik van langere tijdseries. Deze gegevens zijn beschikbaar, maar zijn doorgaans puntmetingen (en dus niet direkt vertaalbaar in cross-sectioneel gemiddelde snelheden). Bovendien zijn dergelijke metingen ruimtelijk schaarser dan dertienuurs metingen. Deze optie zal nader worden bekeken in het kader van het nu te ontwikkelen sedimenttransportmodel.

Figuur 10 – Kwaliteit van de M_2 -waterstand te Vlissingen.



Links: fout in de berekende M_2 -getijamplitude van waterstand (m) als functie van de lengte van de tijdserie die is gebruikt voor harmonische analyse (in dagen).

Rechts: verloop van de amplitude van het verticale M_2 -getij versus gebruikte lengte van de tijdserie.

2.6 Toekomstige ontwikkelingen van het basismodel

Hieronder zal kort worden ingegaan op ontwikkelingen van het basismodel die de komende tijd zullen worden geïmplementeerd, en de mogelijke effecten daarvan op de modelresultaten.

2.6.1 Accurater beschrijving van de bodemruwheid

Zoals in Sect. 2.3 aangegeven moet de bodemruwheid s_f voor de M_2 -stroming enerzijds en de M_0 - en M_4 -waterbeweging anderzijds verschillend worden gekozen. Voor het onderhavige model (waarbij de M_0 - en M_4 -getijsnelheid klein zijn ten opzichte van de M_2 -stroming) is dit in Hoofdstuk 5 nader uitgewerkt. Echter, om een lineaire relatie tussen bodemschuifspanning en bodemsnelheid (Vgl. 4) te rechtvaardigen dient s_f ook van de amplitude van de lokale M_2 -bodemsnelheid af te hangen (Zimmerman, 1982). Om het stromingsprobleem op te lossen dient dan een iteratief oplossingsschema te worden gebruikt waarbij een beschrijving in termen van lineaire differentiaalvergelijkingen niet langer mogelijk is. Een dergelijke iteratieve aanpak zal worden ontwikkeld binnen WL-project 15_039 dat een uitbreiding van het hydrodynamische model beoogt. Door een accuratere beschrijving van de bodemruwheid kan de calibratie van het model mogelijk verbeterd worden.

2.6.2 Nauwkeuriger beschrijving voor ondiepere delen.

Kombergingsbreedte

Een eenvoudige manier om droogval mee te nemen is door naast een stroomvoerende breedte ook een kombergingsbreedte in te voeren. Dit leidt tot modificatie van de massabehoudsvergelijking (Winterwerp en Wang, 2013). Dit zal met name van belang zijn voor de voortplantingssnelheid van het getij, en daarmee de fasering van de verschillende getijcomponenten.

Nauwkeuriger benadering van de oplossing

In Sect. 2.1.4 is aangegeven dat de oplossing voor de waterbeweging wordt verkregen door een successieve benadering van een feitelijk oneindige reeks. Het idee is dat doorgaans enkel de eerste twee termen van deze reeks van belang zijn aangezien de verhouding ϵ van getij-amplitude over bodemdiepte klein is. In ondiepe delen geldt deze aanname echter niet meer: weliswaar zal daar gelden dat $\epsilon < 1$, maar ϵ is niet langer klein. Er dienen dan meer bijdragen aan de reeksontwikkeling voor waterstand en snelheid te worden berekend. Dit kan met name een rol spelen bij de beschrijving van de getijbeweging in de landwaartse, ondiepe delen van het Scheldebekken. Deze aanpak zal nader worden uitgewerkt in bovengenoemd WL-project 15_039.

2.6.3 Turbulentiemodel

In het model wordt tot dusverre gebruik gemaakt van een "partial slip" turbulentiemodel dat gekenmerkt wordt door een constante verticale viscositeit A_v . In de toekomst zal het model worden uitgebreid met een viscositeitsprofiel dat parabolisch afhangt van de hoogte. Dit is realistischer en komt ook meer overeen met de menging zoals die uit complexe numerieke modellen volgt. Dit betekent dat er een betere vergelijking mogelijk zal worden tussen het geïdealiseerde model enerzijds en complexe modellen anderzijds.

2.6.4 Getij-asymmetrie door tijdsafhankelijke verticale menging

In een getijstroming kan door advectie van isohalinen een tijdsafhankelijke verticale stratificatie optreden, welke ertoe leidt dat de verticale menging van sediment en snelheid tijdens eb en vloed verschillend is (Simpson *e.a.*, 1990). Dit mechanisme van Strain Induced Periodic Stratification (SIPS) is een additionele bron van horizontale en verticale getij-asymmetrie die niet in het onderhavige model is meegenomen. Omdat getij-asymmetrie door het SIPS-mechanisme van belang wordt geacht voor het ontstaan van hypertroebelheid zal dit effect ook in het sedimenttransportmodel worden meegenomen.

2.7 Conclusies

Op basis van de modelresultaten kunnen de volgende conclusies worden getrokken:

1. Het model geeft een redelijk goede kwantitatieve beschrijving van de fase van het verticale M_2 -en M_4 -getij.
2. Het model geeft een goede kwalitatieve beschrijving van de amplitude van het verticale M_2 -getij.
3. De amplitude van de M_4 -waterstand is van dezelfde grootte-orde als de gemeten waarden.
4. De amplitudes voor het horizontale M_2 -en M_4 -getij (snelheid) komen qua orde van grootte goed overeen met de gemeten getijcomponenten maar er is geen duidelijke trendmatige overeenkomst. De sterkte van het M_4 -getij in het achterste deel van het bekken wordt door het model overschat.
5. De horizontale getij-asymmetrie zoals berekend door het model komt voor de Westerschelde goed overeen met metingen. In de Zeeschelde voorspelt het model horizontale vloeddominantie, maar metingen laten hier een grote spreiding zien die zowel eb -als vloeddominantie kan impliceren.
6. Het M_4 -snelheidsveld in de Westerschelde wordt overwegend bepaald door het externe M_4 -getij te Vlissingen. De M_4 -stroming in de Boven Zeeschelde is daarentegen het gevolg van de Stokesstroming en de bodemschuifspanning aan het wateroppervlak.
7. Voor wintercondities (hoog debiet) wordt de reststroming gedomineerd door de bijdrage van de rivierafvoer.
8. Voor zomercondities (laag debiet) geeft de Stokes terugkeerstroming de voornaamste bijdrage aan de reststroming, deze is ongeveer tweemaal zo groot als de rivierstroming.
9. Gravitationele circulatie geeft geen belangrijke bijdrage aan de reststroming.

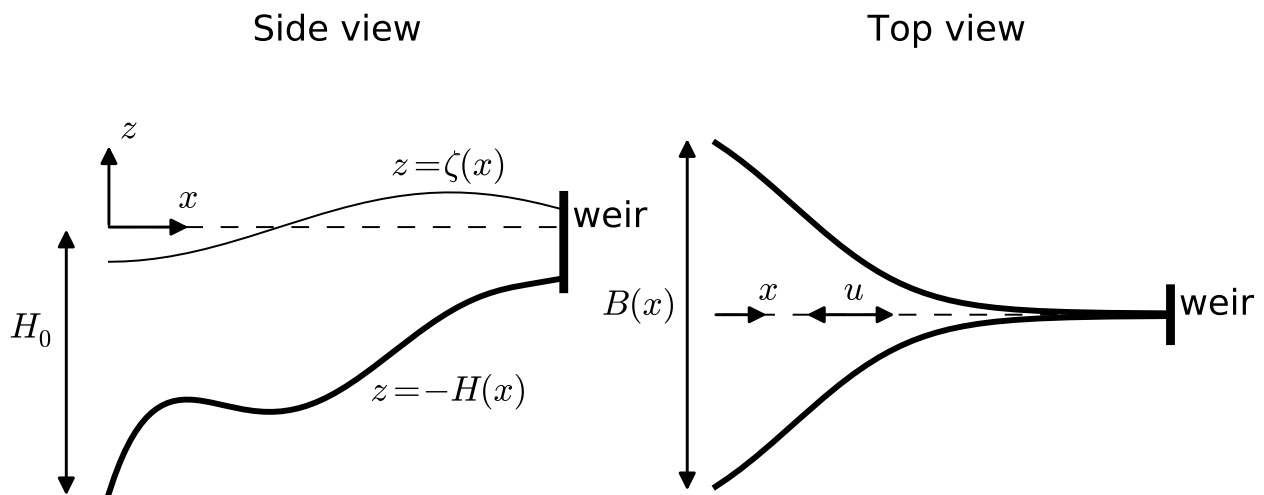
3 Hydrodynamic model

This chapter describes the hydrodynamic model that is part of the idealized sediment transport model that will be developed within WL project 13_103 on hyperturbidity in tidal estuaries. The model is based on an earlier model developed by Schuttelaars and De Swart, (2000) and Chernetsky *et al.*, (2010) and is expanded where appropriate. First, the formulation is introduced and subsequently the solution method and corresponding solutions for the water level and horizontal and vertical flow velocities are given.

3.1 Model formulation

The hydrodynamic two-dimensional, width-averaged (2DV) model calculates elevation $\zeta(x, t)$, horizontal flow velocity $u(x, z, t)$ and vertical flow velocity $w(x, z, t)$ in a straight channel of length L with varying longitudinal bed profile $H(x)$ and channel width $B(x)$ (see Fig. 11). The bottom profile $H(x)$ and channel width $B(x)$ are allowed to vary gradually over the x -direction, i.e. with length scales corresponding to the length of the tidal wave. The sidewalls of the channel are assumed to be vertical and tidal flats are not present. The seaward boundary of the estuary is located at $x = 0$ and the landward boundary is located at $x = L$, where the channel is constrained by a weir with a constant discharge Q . There are no tributaries considered in the model.

Figure 11 – Geometry of the tidal basin as adopted by the idealized model: (a) side view, (b): top view.



It is assumed that the water motion in the estuary is dominated by tidal forcing, with effects of river discharge being relatively small. Moreover, the effect of wind stress and wind waves on the water motion is neglected. Under the above mentioned assumptions, momentum and mass balance is expressed by the width-averaged shallow water equations:

$$u_t + uu_x + wu_z = -g\zeta_x - g \int_z^\zeta \frac{\rho_x}{\rho_0} dz + (A_v u_z)_z, \quad (11)$$

$$u_x + w_z + \frac{B_x}{B} u = 0. \quad (12)$$

In these equations, t is time, g is gravitational acceleration, ρ is density, ρ_0 is a constant reference density and A_v is vertical eddy viscosity coefficient. The subscripts $(\cdot)_x$, $(\cdot)_z$ and $(\cdot)_t$ denote the derivative of a variable in the respective dimension. In Eq. (11), the left-hand side contains inertia (first term) and advection (second and third term). The exerting forces are found on the right-hand side, representing barotropic pressure gradient force, baroclinic pressure gradient force and internal frictional force, respectively.

Eqs. (11) and (12) are subject to several boundary condition. At the free surface $z = \zeta$, the boundary conditions are the no stress condition and the kinematic boundary condition

$$A_v u_z = 0, \quad (\text{no stress}) \quad (13)$$

$$w = \zeta_t + u \zeta_x. \quad (\text{kinematic}) \quad (14)$$

At the bottom $z = -H$, we assume the bed to be impermeable and prescribe a partial slip condition (Schramkowski and Swart, 2002)

$$w = -u H_x, \quad (\text{impermeable bed}) \quad (15)$$

$$A_v u_z = s_f u. \quad (\text{partial slip}) \quad (16)$$

Note that the partial slip condition is evaluated at the top of the constant stress layer instead of at the true bed. Here, the parameter s_f is the so-called stress parameter that can still depend on the longitudinal coordinate. Following Friedrichs and Hamrick, (1996) and Schramkowski and Swart, (2002), this dependency is taken to be linear in the local water depth

$$s_f = s_{f_0} \left(\frac{H(x)}{H_0} \right)^n, \quad (17)$$

where s_{f_0} is constant and n a factor that generally varies between 1 and 3/2, and H_0 is the water depth at the entrance of the estuary. Note that the bottom boundary condition is a linear relation between bed shear stress and velocity, which is a result of a Lorentz linearization procedure (Zimmerman, 1982).

The water motion is forced by a prescribed tidal elevation on the seaside of the estuary at $x = 0$ that consists of a semi-diurnal (M_2) and a M_4 constituent

$$\zeta = A_{M_2} \cos(\sigma t) + A_{M_4} \cos(2\sigma t - \varphi), \quad (18)$$

where A_{M_2} and A_{M_4} are the amplitudes of the M_2 and M_4 tidal constituents, respectively, $\varphi = \varphi_{M_4} - 2\varphi_{M_2}$ the phase difference between the M_2 and M_4 tidal constituents and σ the tidal frequency.

At the riverine side $x = L$, a constant river discharge Q is prescribed and the tidal discharge is required to vanish, so

$$B(x) \int_{-H}^{\zeta} u dz = -Q. \quad (19)$$

The continuity equation (12) will also be used in depth-averaged form to derive the ordinary differential equation for the water level. Using Eqs. (12), (14) and (15), it is given by

$$\zeta_t + \left(\frac{d}{dx} + \frac{B_x}{B} \right) \int_{-H}^{\zeta} u dz = 0. \quad (20)$$

The channel density $\rho(x, z, t)$ of the estuarine water varies in general due to the salinity s , suspended sediment concentration c and temperature T . In this report, we neglect density gradients caused by suspended sediment and temperature and, thus, density only varies due to salinity. The equation of state for the channel density is taken to be linear (Chernetsky *et al.*, 2010), and is given by

$$\rho = \rho_0(1 + \beta_s s), \quad (21)$$

where s is salinity in psu and $\beta_s \sim 7.7 \times 10^{-4} \text{ psu}^{-1}$ a constant that converts salt to density. It is assumed that the salinity is vertically well-mixed, which means that the vertical variations of the salinity field are small compared to its depth- and time-averaged value. Hence, we can write $s \simeq \langle s(x) \rangle$, where angular brackets $\langle \cdot \rangle$ denote a tidal average. In this report, the hydrodynamic model is diagnostic in salinity (hence density), where the expression for longitudinal salinity profile is given by (Talke *et al.*, 2009; Warner *et al.*, 2005)

$$s(x) = \frac{\hat{s}}{2} \left[1 - \tanh \left(\frac{x - x_c}{x_L} \right) \right], \quad (22)$$

where $\hat{s}$ is the salinity at sea in psu, x_c is the location in the estuary in meters where the salinity gradient is largest and x_L is the length scale in meters over which the salinity decays (a measure for the size of the salt wedge).

Under the above mentioned assumptions, the baroclinic pressure gradient term in Eq. (11) can be rewritten as

$$g \int_z^{\zeta} \frac{\rho_x}{\rho_0} dz \simeq g \beta_s \langle s_x \rangle (\zeta - z). \quad (23)$$

Following Friedrichs and Hamrick, 1996, the vertical eddy viscosity coefficient A_v is parameterized as

$$A_v(x) = A_{v_0} \left(\frac{H(x)}{H_0} \right)^m, \quad (24)$$

where A_{v_0} is constant, m is a factor that generally varies between 0 and 3/2. Hence, it is assumed that A_v is independent of height z and can be taken out of the parentheses in Eq. (11). Furthermore, asymmetry in mixing that is a result of time-dependent stratification (Cheng *et al.*, 2010; Stacey *et al.*, 2010, 2001) is neglected.

3.2 Perturbation analysis and solutions

To reveal the relative importance of the various mechanisms that control the water motion in an estuary, the system of equations introduced in the previous section is approximated using a perturbation analysis to obtain (semi-)analytical solutions. In this section, a short outline is given of the procedure used to construct the reduced system of equation and the corresponding solutions. A detailed description of the scaling and the method to solve the system is given in Appendix A1.

First, a scaling analysis is performed to assess the relative importance of the various terms in the model equations. One of the dimensionless parameters that appears in the scaled equations is the ratio of the M_2 tidal amplitude A_{M_2} and the undisturbed water depth H_0 . This ratio, denoted by ϵ , is much smaller than one, i.e. $\epsilon = A_{M_2}/H_0 \ll 1$. The other dimensionless parameters are compared to this small parameter ϵ .

Next, we approximate the solution of the dimensionless equations and the boundary conditions by expanding the physical variables in power series of the small parameter ϵ . This expansion is substituted in the scaled equations and terms of each order of ϵ are collected. This results in a leading order, $\mathcal{O}(\epsilon^0)$, and a first order, $\mathcal{O}(\epsilon^1)$, system of equations.

3.2.1 Leading order system of equations and solutions

At leading order, the dimensional system of equations describing the water motion reads

$$A_v u_{zz}^0 - u_t^0 = g \zeta_x^0, \quad (25)$$

$$u_x^0 + w_z^0 + \frac{B_x}{B} u^0 = 0, \quad (26)$$

$$\zeta_t^0 + \left(\frac{d}{dx} + \frac{B_x}{B} \right) \int_{-H}^0 u^0 dz = 0. \quad (27)$$

The superscript denotes the order of ϵ . Note that in the momentum equation (25) at leading order, the advection terms uu_x and uw_z do not reappear. Additionally, the assumption that the horizontal density gradient is small has the consequence that the baroclinic pressure is of order ϵ and is not present in Eq. (25).

The corresponding boundary conditions read

$$\zeta^0 = A_{M_2} \cos(\sigma t), \quad \text{at } x = 0, \quad (28)$$

$$\int_{-H}^0 u^0 dz = 0, \quad \text{at } x = L, \quad (29)$$

$$w^0 = \zeta_t^0, \quad \text{at } z = 0, \quad (30)$$

$$A_v u_z^0 = 0, \quad \text{at } z = 0, \quad (31)$$

$$A_v u_z^0 = s_f u^0, \quad \text{at } z = -H, \quad (32)$$

$$w^0 = -u^0 H_x, \quad \text{at } z = -H. \quad (33)$$

The solution to Eqs. (25)-(33) describes the propagation of a tidal wave in a medium with effectively a homo-

geneous density and a varying depth and width. They are solved by assuming solutions of the form

$$(u^0, w^0, \zeta^0) = \Re \left\{ \left(\hat{u}^0(x, z), \hat{w}^0(x, z), \hat{\zeta}^0(x, z) \right) e^{i\sigma t} \right\}, \quad (34)$$

where $\Re\{\cdot\}$ denotes the real part of the expression between braces and i is the imaginary unit. $\hat{u}^0(x, z)$, $\hat{w}^0(x, z)$ and $\hat{\zeta}^0(x, z)$ are the complex amplitudes of the spatial variables. Next, the momentum equation (25) is solved resulting in an expression for $\hat{u}^0$ as a function of the water level $\hat{\zeta}^0$

$$\hat{u}^0 = \frac{g\hat{\zeta}_x^0}{i\sigma} \left(\alpha_{M_2} \cosh(r_{M_2} z) - 1 \right), \quad (35)$$

with

$$\alpha_{M_2}(x) = \frac{s_f}{\left(A_v r_{M_2} \sinh(r_{M_2} H) + s_f \cosh(r_{M_2} H) \right)}, \quad (36)$$

$$r_{M_2}(x) = \sqrt{\frac{i\sigma}{A_v}}. \quad (37)$$

Subsequently, substituting Eq. (35) into the depth-averaged continuity equation (27) leads to a second order linear ODE for $\hat{\zeta}^0$

$$T_1 \hat{\zeta}_{xx}^0 + T_2 \hat{\zeta}_x^0 - T_3 \hat{\zeta}^0 = 0, \quad (38)$$

with

$$T_1 = \frac{\alpha_{M_2}}{r_{M_2}} \sinh(r_{M_2} H) - H, \quad (39)$$

$$T_2 = \frac{B_x}{B} T_1 - \frac{H_x r_{M_2}^2 + (\alpha_{M_2} r_{M_2, x} - \alpha_{M_2, x} r_{M_2}) \sinh(r_{M_2} H) - \alpha_{M_2} r_{M_2} (H_x r_{M_2} + H r_{M_2, x}) \cosh(r_{M_2} H)}{r_{M_2}^2}, \quad (40)$$

$$T_3 = \frac{\sigma^2}{g}. \quad (41)$$

This equation generally needs to be solved numerically because of the non-constant coefficients in the ODE. Finally, we find the solution for $\hat{w}^0$ from the continuity equation (26)

$$\hat{w}^0 = T_1' \hat{\zeta}_{xx}^0 - T_2' \hat{\zeta}_x^0 - T_3 \hat{\zeta}^0, \quad (42)$$

with

$$T_1' = \frac{\alpha_{M_2}}{r_{M_2}} \sinh(r_{M_2} z) + z, \quad (43)$$

$$T_2' = \frac{B_x}{B} T_1' + \frac{\alpha_{M_2, x} r_{M_2} - \alpha_{M_2} r_{M_2, x}}{r_{M_2}^2} \sinh(r_{M_2} z) + \frac{\alpha_{M_2} r_{M_2, x}}{r_{M_2}} z \cosh(r_{M_2} z). \quad (44)$$

$$(45)$$

A special case in which an analytical solution for $\hat{\zeta}^0$ can be found is when α , r_{M_2} , H and B_x/B are uniform in the x -direction. Eq. (38) then reduces to

$$T_1 \hat{\zeta}_{xx}^0 + \frac{B_x}{B} T_1 \hat{\zeta}_x^0 - T_3 \hat{\zeta}^0 = 0, \quad (46)$$

and the solution to this equation is

$$\hat{\zeta}^0 = \left[\frac{\Gamma \cosh(\beta\Gamma(x-1)) + \sinh(\beta\Gamma(x-1))}{\Gamma \cosh(\beta\Gamma) - \sinh(\beta\Gamma)} \right] e^{-\beta x}, \quad (47)$$

with

$$\beta = \frac{1}{2} \frac{B_x}{B}, \quad (48)$$

$$\Gamma = \sqrt{1 + \frac{\sigma^2}{g\beta^2 \left(\frac{\alpha_{M_2}}{r_{M_2}} \sinh(r_{M_2} H) - H \right)}}. \quad (49)$$

Subsequently, for this special case, the expressions for $\hat{u}^0$, Eq. (35), and $\hat{w}^0$, Eq. (42) can be rewritten using the expression for $\hat{\zeta}^0$, Eq. (47).

3.2.2 First order system of equations and solutions

At first order, $\mathcal{O}(\epsilon^1)$, the dimensional momentum and continuity equations are given by

$$A_v u_{zz}^1 - u_t^1 = g\zeta_x^1 + \xi - g\beta_s \langle s_x \rangle z, \quad (50)$$

$$u_x^1 + w_x^1 + \frac{B_x}{B} u^1 = 0, \quad (51)$$

$$\zeta_t^1 + \left(\frac{d}{dx} + \frac{B_x}{B} \right) \left(\int_{-H}^0 u^1 dz + \gamma \right) = 0, \quad (52)$$

where we have introduced the following simplifying notations

$$\xi(x, z, t) = u^0(x, z, t)u_x^0(x, z, t) + w^0(x, z, t)u_z^0(x, z, t), \quad (53)$$

$$\gamma(x, t) = \zeta^0(x, t)u^0(x, t)|_{z=0}, \quad (54)$$

$$\chi(x, t) = \zeta^0(x, t)u_{zz}^0(x, t)|_{z=0}. \quad (55)$$

Here, ξ represents advection of momentum and γ is related to the tidal return flow or Stokes return flow, which is the result of a positive correlation between the zeroth order vertical and horizontal tide. Finally, χ originates from the first order contribution of the stress free boundary condition.

The boundary conditions for the system of equations at first order read

$$\zeta^1 = A_{M_4} \cos(2\sigma t - \varphi), \quad \text{at } x = 0, \quad (56)$$

$$\int_{-H}^0 u^1 dz = -\frac{Q}{B} - \gamma, \quad \text{at } x = L, \quad (57)$$

$$w^1 + w_z^0 \zeta^0 = \zeta_t^1 + u^0 \zeta_x^0, \quad \text{at } z = 0, \quad (58)$$

$$A_v u_z^1 = -A_v \chi, \quad \text{at } z = 0, \quad (59)$$

$$A_v u_z^1 = s_f u^1, \quad \text{at } z = -H, \quad (60)$$

$$w^1 = -u^1 H_x, \quad \text{at } z = -H. \quad (61)$$

As is apparent from Eq. (50)-(61), the first order velocity and water level are forced externally by an M_4 tidal component and a constant river discharge and internally by the salinity gradient, the leading order advection, and tidal return flow (Stokes return flow). Since we assumed solutions with a M_2 tidal frequency in Section 3.2.1, Eq. (34), the product of two forcing terms with that M_2 frequency result in a residual (M_0) and a M_4 frequency (see Appendix A1.2.2). The general solution is of the form

$$\Phi^1 = \Phi^{10} + \Phi^{14}, \quad (62)$$

where $\Phi = (\zeta, u, w)$ and the first and second superscript denote the order of ϵ and the tidal component, respectively. In Eq. (62), Φ^{10} denotes the tidally averaged part and Φ^{14} is given by

$$\Phi^{14} = \Re \left\{ \hat{\Phi}^{14} e^{2i\sigma t} \right\} \quad (63)$$

The equations and solutions for both frequencies are given below.

Residual flow velocity (M_0)

The equations for the residual flow are obtained by taking the tide-averaged component of the first order equations. Let $\langle \cdot \rangle$ denote tide-averaging, the momentum equation with its boundary conditions is then given by

$$A_v \hat{u}_{zz}^{10} = g \hat{\zeta}_x^{10} + \langle \hat{\xi} \rangle - g \beta_s \langle \hat{s}_x \rangle z, \quad (64)$$

$$\hat{\zeta}^{10} = 0, \quad \text{at } x = 0, \quad (65)$$

$$\int_{-H}^0 \hat{u}^{10} dz = -\frac{Q}{B} - \langle \hat{\gamma} \rangle, \quad \text{at } x = L, \quad (66)$$

$$A_v \hat{u}_z^{10} = -A_v \langle \hat{\chi} \rangle, \quad \text{at } z = 0, \quad (67)$$

$$A_v \hat{u}_z^{10} = s_f \hat{u}^{10}, \quad \text{at } z = -H, \quad (68)$$

where the first number in the superscript refers to the order and the second number to the frequency. Since the solution to $\hat{u}^{10}$ is linear, it can be constructed by adding the contributions of the different forcing terms to the residual velocity, i.e.,

$$\hat{u}^{10} = u^{\text{salt}} + u^{\text{adv}} + u^{\text{no-stress}} + u^{\text{river}} + u^{\text{stokes}}. \quad (69)$$

The expressions for the contribution of each forcing term to the residual velocity are derived by taking into account only the terms in the momentum equation and appropriate boundary conditions for each contribution. Here, only the resulting expressions are given. For a detailed analysis, the reader is referred to Appendix A1.

Salinity field: The contribution of the salinity field to the residual flow velocity is

$$u^{\text{salt}} = \left[\frac{1}{2} \frac{z^2}{A_v} - \frac{1}{2} \frac{H^2}{A_v} - \frac{H}{s_f} \right] g \hat{\zeta}_x^{\text{salt}} - \left[\frac{1}{6} \frac{z^3}{A_v} + \frac{1}{6} \frac{H^3}{A_v} + \frac{H^2}{2s_f} \right] g \beta_s \langle \hat{s}_x \rangle, \quad (70)$$

with

$$\hat{\zeta}_x^{\text{salt}} = - \frac{\beta_s \langle \hat{s}_x \rangle \left[\frac{1}{8} \frac{H^4}{A_v} + \frac{1}{2} \frac{H^3}{s_f} \right]}{\left[\frac{1}{3} \frac{H^3}{A_v} + \frac{H^2}{s_f} \right]}. \quad (71)$$

This contribution is also referred to as gravitational circulation.

Advection of momentum: The contribution of the advection of momentum to the residual flow velocity is

$$u^{\text{adv}} = \left[\frac{1}{2} \frac{z^2}{A_v} - \frac{1}{2} \frac{H^2}{A_v} - \frac{H}{s_f} \right] g \hat{\zeta}_x^{\text{adv}} - \frac{1}{A_v} \left[z + H + \frac{A_v}{s_f} \right] \int_{-H}^0 \langle \hat{\xi} \rangle dz + \frac{1}{A_v} \left[z \int_{-H}^z \langle \hat{\xi} \rangle dz' - \int_{-H}^z \langle \hat{\xi} \rangle z' dz' \right], \quad (72)$$

with

$$\hat{\zeta}_x^{\text{adv}} = \frac{\frac{1}{A_v} \int_{-H}^0 \left[z \int_{-H}^z \langle \hat{\xi} \rangle dz' - \int_{-H}^z \langle \hat{\xi} \rangle z' dz' \right] dz - \left[\frac{1}{2} \frac{H^2}{A_v} + \frac{H}{s_f} \right] \int_{-H}^0 \langle \hat{\xi} \rangle dz}{g \left[\frac{1}{3} \frac{H^3}{A_v} + \frac{H^2}{s_f} \right]}. \quad (73)$$

Stress free boundary condition: The contribution of the stress free boundary condition to the residual flow velocity is

$$u^{\text{no-stress}} = \left[\frac{1}{2} \frac{z^2}{A_v} - \frac{1}{2} \frac{H^2}{A_v} - \frac{H}{s_f} \right] g \hat{\zeta}_x^{\text{no-stress}} - \left[z + H + \frac{A_v}{s_f} \right] \langle \hat{\chi} \rangle, \quad (74)$$

with,

$$\hat{\zeta}_x^{\text{no-stress}} = - \frac{\langle \hat{\chi} \rangle \left[\frac{H^2}{2} + \frac{A_v H}{s_f} \right]}{g \left[\frac{1}{3} \frac{H^3}{A_v} + \frac{H^2}{s_f} \right]}. \quad (75)$$

River outflow: The contribution of the river outflow to the residual flow velocity is

$$u^{\text{river}} = \left[\frac{1}{2} \frac{z^2}{A_v} - \frac{1}{2} \frac{H^2}{A_v} - \frac{H}{s_f} \right] g \hat{\zeta}_x^{\text{river}}, \quad (76)$$

with,

$$\hat{\zeta}_x^{\text{river}} = \frac{Q}{gB \left[\frac{1}{3} \frac{H^3}{A_v} + \frac{H^2}{s_f} \right]}. \quad (77)$$

Stokes return flow: The contribution of the Stokes return flow to the residual flow velocity is

$$u^{\text{stokes}} = \left[\frac{1}{2} \frac{z^2}{A_v} - \frac{1}{2} \frac{H^2}{A_v} - \frac{H}{s_f} \right] g \hat{\zeta}_x^{\text{stokes}}, \quad (78)$$

with,

$$\hat{\zeta}_x^{\text{stokes}} = \frac{\langle \hat{\gamma} \rangle}{g \left[\frac{1}{3} \frac{H^3}{A_v} + \frac{H^2}{s_f} \right]}. \quad (79)$$

The residual (constant) water level associated with each contribution can be obtained by numerically integrating the water level gradients with respect to x .

M₄ flow velocity

The equations for the M₄ flow velocity are obtained by taking the M₄ component of a signal, denoted by $[\cdot]$ (see Appendix A1 for further details). The momentum equation with its boundary conditions is then given by

$$A_v \hat{u}_{zz}^{14} - 2i\sigma \hat{u}^{14} = g \hat{\zeta}_x^{14} + 2[\hat{\xi}], \quad (80)$$

$$\hat{\zeta}^{14} = A_{M_4} e^{-i\varphi}, \quad \text{at } x = 0, \quad (81)$$

$$\int_{-H}^0 \hat{u}^{14} dz = -2[\hat{\gamma}], \quad \text{at } x = L, \quad (82)$$

$$A_v \hat{u}_z^{14} = -2A_v [\hat{\chi}], \quad \text{at } z = 0, \quad (83)$$

$$A_v \hat{u}_z^{14} = s_f \hat{u}^{14}, \quad \text{at } z = -H. \quad (84)$$

$$(85)$$

Note that, in contrast to the residual velocity, the M₄ flow velocity does not contain the forcing terms due to the river outflow and the salinity field. The solution to $\hat{u}^{14}$ can be constructed by adding the contributions of the different forcing terms to the M₄ flow velocity, i.e.

$$\hat{u}^{14} = u^{\text{tide}} + u^{\text{adv}} + u^{\text{no-stress}} + u^{\text{stokes}}. \quad (86)$$

The expressions for the flow velocities for each contribution together with the associated water levels are given below.

External M_4 tide: The contribution of the external M_4 tide to the M_4 flow velocity is similar to the solution derived for the leading order flow velocity. It reads

$$u^{\text{tide}} = \frac{g\hat{\zeta}_x^{\text{tide}}}{2i\sigma} (\alpha_{M_4} \cosh(r_{M_4} z) - 1), \quad (87)$$

with

$$\alpha_{M_4} = \frac{s_f}{(A_v r_{M_4} \sinh(r_{M_4} H) + s_f \cosh(r_{M_4} H))}, \quad (88)$$

$$r_{M_4} = \sqrt{\frac{2i\sigma}{A_v}}. \quad (89)$$

The complex water level amplitude $\hat{\zeta}^{\text{tide}}$ due to the external tide is obtained by substituting the expression for u^{tide} into the depth-integrated continuity equation (52). What follows is a second order ordinary differential equation (ODE) that needs to be solved numerically,

$$F_1 \hat{\zeta}_{xx}^{\text{tide}} + F_2 \hat{\zeta}_x^{\text{tide}} - F_3 \hat{\zeta}^{\text{tide}} = 0, \quad (90)$$

with

$$F_1 = \frac{\alpha_{M_4}}{r_{M_4}} \sinh(r_{M_4} H) - H, \quad (91)$$

$$F_2 = \left(\frac{d}{dx} + \frac{B_x}{B} \right) F_1, \quad (92)$$

$$F_3 = \frac{4\sigma^2}{g}, \quad (93)$$

Advection of momentum: The contribution of advection of momentum to the M_4 flow velocity is

$$u^{\text{adv}} = \frac{g\hat{\zeta}_x^{\text{adv}}}{2i\sigma} (\alpha_{M_4} \cosh(r_{M_4} z) - 1) + \frac{G_1}{A_v r_{M_4}} - \frac{\alpha_{M_4} G_2}{A_v r_{M_4} s_f} (A_v r_{M_4} \cosh(r_{M_4} (z + H)) + s_f \sinh(r_{M_4} (z + H))), \quad (94)$$

with

$$G_1 = e^{r_{M_4} z} \int_{-H}^z [\hat{\xi}] e^{-r_{M_4} z'} dz' - e^{-r_{M_4} z} \int_{-H}^z [\hat{\xi}] e^{r_{M_4} z'} dz', \quad (95)$$

$$G_2 = \int_{-H}^0 [\hat{\xi}] e^{-r_{M_4} z} dz + \int_{-H}^0 [\hat{\xi}] e^{r_{M_4} z} dz. \quad (96)$$

The ODE for the corresponding water level reads

$$F_1 \hat{\zeta}_{xx}^{\text{adv}} + F_2 \hat{\zeta}_x^{\text{adv}} - F_3 \hat{\zeta}^{\text{adv}} = F_{\text{adv}}, \quad (97)$$

with,

$$F_{\text{adv}} = \frac{A_v r_{M_4}^2}{g} \left(\frac{d}{dx} + \frac{B_x}{B} \right) \left(\frac{G_2}{A_v r_{M_4}^2} (1 - \alpha_{M_4}) - \frac{1}{A_v r_{M_4-H}} \int_0^0 G_1 dz \right), \quad (98)$$

Stress free boundary condition: The contribution of the stress free boundary condition to the M_4 flow velocity is

$$u^{\text{no-stress}} = \frac{g \hat{\zeta}_x^{\text{no-stress}}}{2i\sigma} (\alpha_{M_4} \cosh(r_{M_4} z) - 1) - \frac{2\alpha_{M_4}}{r_{M_4} s_f} [\hat{\chi}] \left(A_v r_{M_4} \cosh(r_{M_4} (z + H)) + s_f \sinh(r_{M_4} (z + H)) \right). \quad (99)$$

The ODE for the corresponding water level reads

$$F_1 \hat{\zeta}_{xx}^{\text{no-stress}} + F_2 \hat{\zeta}_x^{\text{no-stress}} - F_3 \hat{\zeta}^{\text{no-stress}} = F_{\text{no-stress}}, \quad (100)$$

with

$$F_{\text{no-stress}} = \frac{2i\sigma}{g} \left(\left(\frac{B_x}{B} [\hat{\chi}] + [\hat{\chi}]_x \right) \frac{(1 - \alpha_{M_4})}{r_{M_4}^2} - \frac{[\hat{\chi}] (r_{M_4} \alpha_{M_4,x} + 2r_{M_4,x} (1 - \alpha_{M_4}))}{r_{M_4}^3} \right), \quad (101)$$

Stokes return flow: The contribution of the Stokes return flow to the M_4 flow velocity is

$$u^{\text{stokes}} = \frac{g \hat{\zeta}_x^{\text{stokes}}}{2i\sigma} (\alpha_{M_4} \cosh(r_{M_4} z) - 1). \quad (102)$$

The ODE for the corresponding water level reads

$$F_1 \hat{\zeta}_{xx}^{\text{stokes}} + F_2 \hat{\zeta}_x^{\text{stokes}} - F_3 \hat{\zeta}^{\text{stokes}} = F_{\text{stokes}}, \quad (103)$$

with

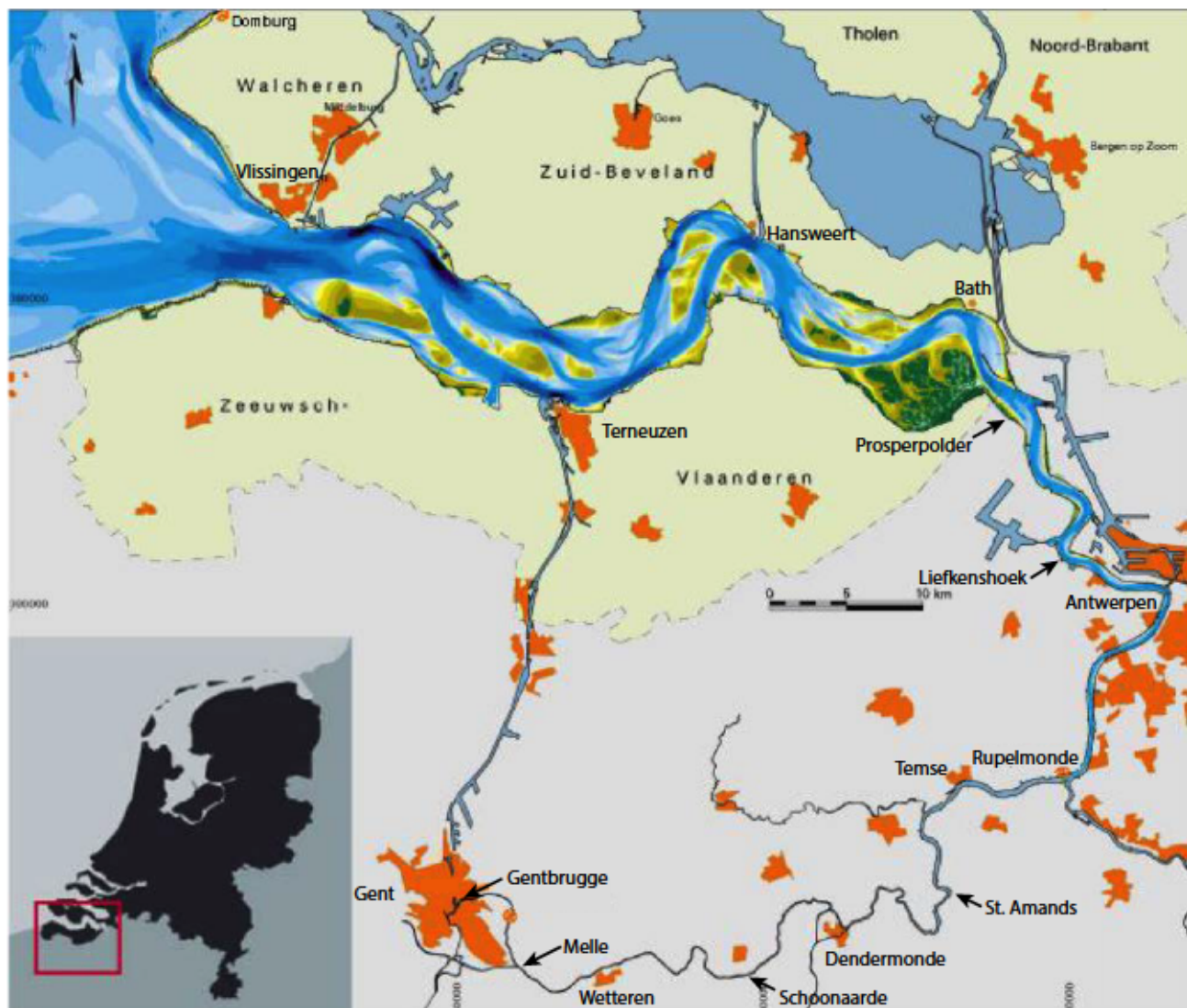
$$F_{\text{stokes}} = -\frac{4i\sigma}{g} \left([\hat{\gamma}]_x + \frac{B_x}{B} [\hat{\gamma}] \right). \quad (104)$$

4 Data Scheldt Estuary

The idealized hydrodynamic model described in the previous section is used to reproduce the water motion in the Scheldt Estuary as close as possible. In this section, the data used for the formulation of the model schematization, boundary conditions and model calibration are presented.

4.1 Model domain

Figure 12 – The Scheldt Estuary.



The model describes the Scheldt Estuary from Vlissingen to Gentbrugge, which has a total length of 160 km (Fig. 12). The Scheldt Estuary can be divided into three sections, the Western Scheldt (Dutch part) and the Lower and Upper Sea Scheldt (Belgian part). The transitions are situated at the Dutch-Belgian border (54 km from Vlissingen) for the Western Scheldt and the Lower Sea Scheldt, and at Rupelmonde (89 km from Vlissingen) for the Lower and the Upper Sea Scheldt.

4.2 Bottom and width profile

The model uses a one-dimensional bottom $H(x)$ and width $B(x)$ profile. To determine these profiles, use is made of an earlier study with the 2DH model of Coen *et al.*, (2016). The width-averaged bottom is derived from this model using many channel sections defined throughout the Scheldt Estuary. These sections were used previously in a study by Plancke *et al.*, (2014). For every section, the wetted cross-sectional area and width are determined at local high and low water level. Subsequently, the width profile $B(x)$ for mean water level follows from averaging the width for local high and low water level. The width-averaged bottom profile $H(x)$ is obtained by taking the ratio of the wetted cross-sectional area and the width for each section. Both the bottom and width profile are presented in Fig. 13. It turns out that these profiles are still jagged. Since the idealized model only allows the channel bottom and width to vary gradually over the x -direction, a polynomial is fitted through the data (see Fig. 13). For the width profile, the following polynomial was used

$$p_B(x) = \exp\left(\frac{k_1x + k_2}{k_3x^2 + k_4x + k_5}\right), \quad (105)$$

with coefficients $k_1 = -2.742 \times 10^{-5}$, $k_2 = 1.8973$, $k_3 = 4.9788 \times 10^{-11}$, $k_4 = -9.213 \times 10^{-6}$, $k_5 = 1$, and x in m. For the depth profile, a fifth order polynomial is used

$$p_H(x) = d_1x^5 + d_2x^4 + d_3x^3 + d_4x^2 + d_5x + d_6, \quad (106)$$

with coefficients $d_1 = -2.9013 \times 10^{-24}$, $d_2 = 1.4030 \times 10^{-18}$, $d_3 = -2.4218 \times 10^{-13}$, $d_4 = 1.7490 \times 10^{-8}$, $d_5 = -5.2141 \times 10^{-4}$, $d_6 = 15.332$, and x in m.

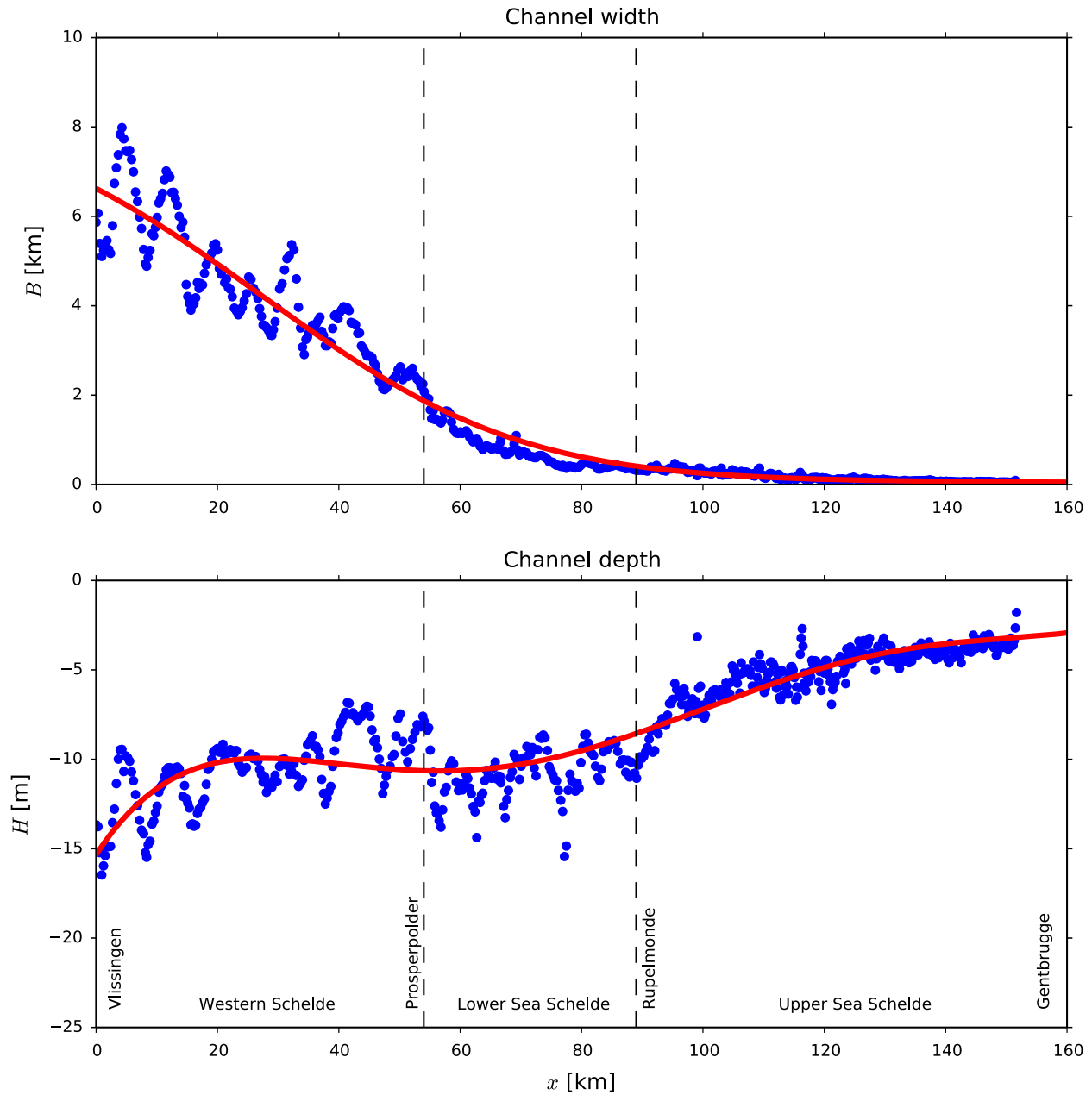
4.3 Water levels

Table 5 – Water level stations in the Scheldt Estuary.

Name station	Distance to Vlissingen (km)	Section
Vlissingen	0	Western Scheldt
Terneuzen	18.5	Western Scheldt
Hansweert	33.8	Western Scheldt
Bath	49.8	Western Scheldt
Prosperpolder	54	Lower Sea Scheldt
Liefkenshoek	61.1	Lower Sea Scheldt
Antwerpen	75.6	Lower Sea Scheldt
Temse	97.3	Upper Sea Scheldt
St. Amands	106.8	Upper Sea Scheldt
Dendermonde	119.8	Upper Sea Scheldt
Schoonaarde	130.6	Upper Sea Scheldt
Wetteren	142.7	Upper Sea Scheldt
Melle	148.8	Upper Sea Scheldt

The water levels used for the boundary conditions and the calibration are those from the year 2013 and are known with intervals of 10 minutes. The data for the Western Scheldt and the Sea Scheldt are provided by

Figure 13 – Channel width (top panel) and depth (bottom panel) of the Scheldt Estuary in 2013.



Blue dots denote measurements and red lines correspond to the best fit according to Eqs. (105) and (106) for the width and depth, respectively.

Hydro Meteo Centrum Zeeland (HMCZ) and Hydrologisch InformatieCentrum (HIC), respectively. Use is made of 12 water level stations, which are listed in Table 5.

Based on the time series of the water level at each station, a harmonic analysis is performed using the Matlab program `t_tide` to obtain the M_2 and M_4 tidal components (see Table 6). Based on Table 6, this implies that the model is forced at the open boundary (Vlissingen) according to Eq. (18), with $A_{M_2} = 1.77$ m, $A_{M_4} = 0.14$ m and $\varphi = -1.3^\circ$.

Table 6 – Amplitude and phase for the vertical M_2 and M_4 tide at the water level stations in the Scheldt Estuary.

Name station	M_2 amplitude (m)	M_2 phase (°)	M_4 amplitude (m)	M_4 phase (°)
Vlissingen	1.77	0	0.14	-1.3
Terneuzen	1.98	10.4	0.12	12.3
Hansweert	2.03	20.5	0.11	39.4
Bath	2.18	31.1	0.11	58.7
Prosperpolder	2.19	33.1	0.12	62.2
Liefkenshoek	2.26	35.6	0.12	65.1
Antwerpen	2.31	44.3	0.13	74.6
Temse	2.28	62.7	0.16	88.2
St. Amands	2.22	73.4	0.24	101.7
Dendermonde	1.69	93.3	0.25	128.7
Schoonaarde	1.31	116.4	0.24	164.3
Wetteren	1.09	143.4	0.21	212.6
Melle	1.02	157.4	0.22	242.9

Phases are with respect to M_2 high tide in Vlissingen.

4.4 Flow velocities

A comparison has been made between the computed and the measured M_2 and M_4 components of width averaged tidal water motion. The temporal width averaged velocities are taken from the dataset used by Nnafie *et al.*, (2016) who have used data from eleven measurement campaigns along transects. From the two-dimensional horizontal velocity components the along-estuary velocity was obtained by projection along the curve that defines that distance to Vlissingen.

The location and time of the ADCP campaigns are listed in Table 7.

Table 7 – Location and time of velocity measurement campaigns in the Scheldt Estuary.

Location	Distance to Vlissingen (km)	Start time	End time
Terneuzen	17	06-Jul-2011 05:38:19	06-Jul-2011 18:10:22
Gat van Ossensisse	28	09-May-2012 05:01:36	09-May-2012 17:47:13
Schaar van de Waarde	40	24-Apr-2013 05:39:17	24-Apr-2013 18:26:28
Zuidergat	40	25-Apr-2013 06:22:56	25-Apr-2013 19:09:42
Liefkenshoek	63	25-Jun-2013 06:29:45	25-Jun-2013 19:16:03
Oosterweel	74	27-Jun-2013 06:26:47	27-Jun-2013 19:03:47
Kruikeke	85	30-May-2013 06:11:14	30-May-2013 19:06:25
Driegoten	102	12-Jun-2013 06:54:02	12-Jun-2013 19:14:04
Dendermonde	120	17-Apr-2014 07:50:51	17-Apr-2014 19:50:00
Schoonaarde	132	27-May-2013 08:18:12	27-May-2013 20:05:01
Schellebelle	140	15-Apr-2014 07:58:24	15-Apr-2014 19:10:29

From the width averaged longitudinal velocity the corresponding residual, M_2 and M_4 components have been determined by adopting a least square fit with these three constituents. In appendix A2 the adopted least

square fit is briefly discussed and the comparison with the measured along-channel velocity is shown for each location. Based on visual inspection the agreement between velocity measurements and least square approximation appears to be reasonable. Only for the locations Kruike and Driegoten the velocity signal is too noisy for an accurate fit.

4.5 Salinity profile

The tidally averaged salinity profile throughout the Scheldt Estuary is determined based on conductivity measurements carried out in February and October 2009. The measurements in February were performed during high water slack on February 4th and during low water slack on February 10th. Based on these conductivity measurements, the salinity is determined using the UNESCO formulation (Fofonoff and Millard, 1983). A water temperature of 7° C is assumed according to data found in Fettweis *et al.*, (1997). Subsequently, a tidally averaged salinity for each station is obtained by averaging the values of 4 and 10 February. Similarly, the salinity at each station in October is determined based on measurements during low water slack on October 5th and during high water slack on October 13th. Here, a water temperature of 19° C is assumed (Fettweis *et al.*, 1997).

The salinity in February and October 2009 represent winter and summer conditions, respectively (Vandenbruwaene *et al.*, 2016b). To use the salinity data in the model, the tidally averaged profiles were fitted using Eq. (22). The corresponding values for the parameters $\hat{s}$, x_c and x_L for winter and summer conditions are listed in Table 8. The measured tidally averaged salinity and corresponding best fit are shown in Fig. 14.

Table 8 – Parameter values for the best fit for the salinity profile of the Scheldt Estuary.

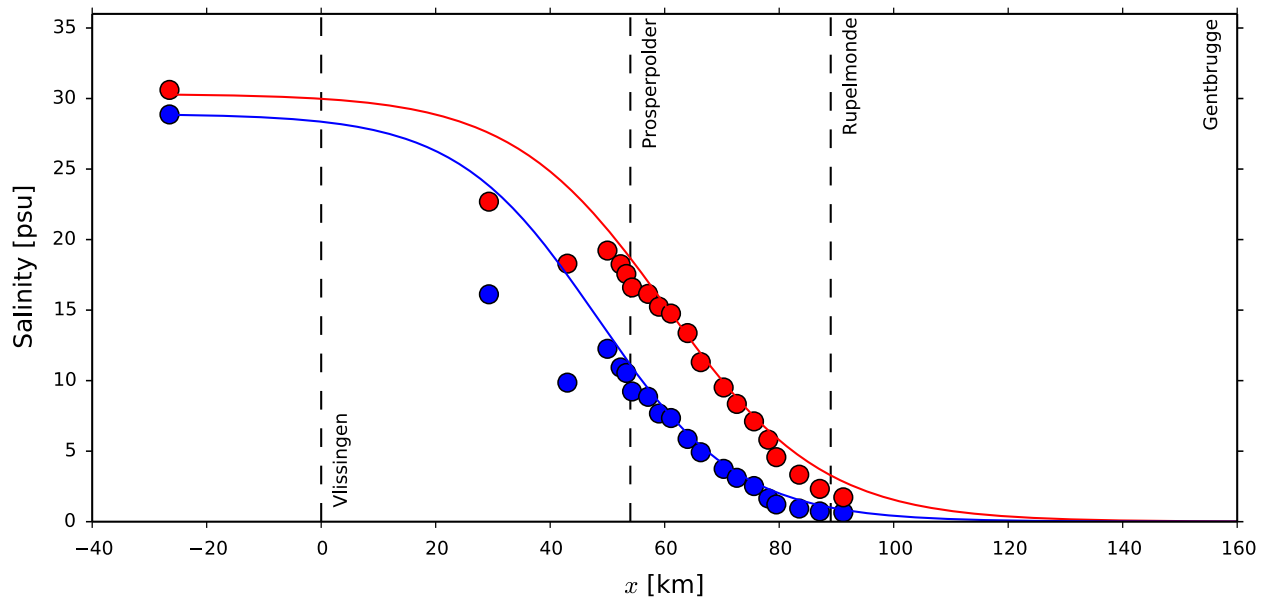
Parameter	Winter	Summer
$\hat{s}$ (psu)	28.9131	30.3296
x_c (km)	48.2068	60.4140
x_L (km)	24.5097	27.1020

4.6 River discharge

At the landward boundary (Gentbrugge) a river discharge needs to be prescribed. The measured daily averaged discharge at Melle (approximately 2 km downstream of Gentbrugge) is shown in Fig. 15. The moments of the salinity measurements in February and October are also shown. It follows that the salinity measurements in February were performed during a period with high river discharges, whereas in October low river discharges were measured.

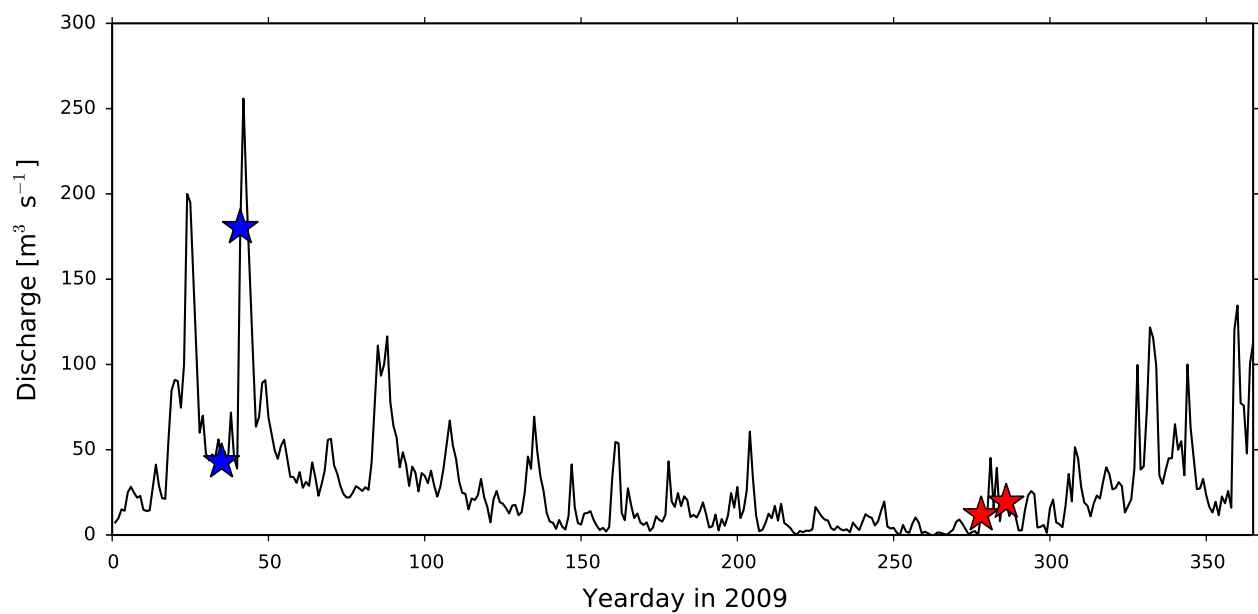
For the model, a constant river discharge needs to be imposed. Ideally, this discharge is consistent with the measured salinity distribution. Therefore, for the constant river discharge a distinction is made between winter (February) and summer conditions (October). This results in a constant discharge in winter of approximately $110 \text{ m}^3 \text{ s}^{-1}$ (the average of the two salinity data in February) and in summer of approximately $15 \text{ m}^3 \text{ s}^{-1}$ (the average of the two salinity data in October).

Figure 14 – Tidally averaged salinity throughout the Scheldt Estuary.



Blue and red dots denote measured values for summer (blue) and winter (red) conditions, respectively. Blue and red lines correspond to the best fit according to Eq. (22) for summer (blue) and winter (red) conditions. The area with negative x -values is not part of the model domain.

Figure 15 – Daily averaged discharge in Melle during 2009.



The dates of the salinity measurements in February and October are denoted with blue and red stars, respectively.

5 Calibration

The idealized model introduced in Section 3 is calibrated based on the data of the Scheldt Estuary discussed in the previous chapter. This calibration is done by optimizing the constant turbulence parameters vertical eddy viscosity, A_{v_0} , and stress parameter, s_{f_0} , such that the observed vertical M_2 and M_4 tides are optimally reproduced for both amplitude and phase. Here, the depth dependence of A_v and s_f has not been considered, i.e. $m = n = 0$.

In practice, the calibration involves minimizing the following cost function (Jones and Davies, 1996)

$$f_{\text{cost}} = \sqrt{(\Delta\hat{\zeta})^2 + 2\hat{\zeta}^{\text{obs}}\hat{\zeta}^{\text{mod}}(1 - \cos \Delta\varphi)}. \quad (107)$$

Here, the superscripts $(.)^{\text{obs}}$ and $(.)^{\text{mod}}$ denote the observed and modeled value of that quantity, respectively. Furthermore, $\Delta\hat{\zeta} = \hat{\zeta}^{\text{obs}} - \hat{\zeta}^{\text{mod}}$ and $\Delta\varphi = \varphi^{\text{obs}} - \varphi^{\text{mod}}$.

It turns out that the calibration is an ambiguous exercise. Both the M_2 and M_4 component have their own cost function that should be minimized. In general, a satisfactory fit for the M_2 water level leads to a worse fit for the M_4 water level and vice versa. In particular, the amplitudes of the M_4 water level are strongly overestimated. To alleviate this issue, the boundary condition at the bottom, Eq. (16) is adapted. This boundary condition assumes a linear relationship between the shear stress and the flow velocity at the bottom. Moreover, this condition is the same for each tidal component, i.e. $A_v u_{M_2, z} = s_f u_{M_2}$, $A_v u_{M_4, z} = s_f u_{M_4}$, etc. Hence, the stress parameter s_f is independent of the tidal component, which is not justified a priori. According to, e.g., Godin, (1999), residual (M_0) and M_4 tidal flow require a larger value of the stress parameter than the M_2 tidal flow. This can be illustrated by considering the boundary condition (16) which is actually a linearisation of the more commonly adopted relation

$$A_v u_z = c_d |u| u. \quad (108)$$

This equation expresses the fact that the bed shear stress is a non-linear function of velocity. If we now use the expansion for the bottom velocity, i.e. $u \approx u^0 + u^1$, we can write Eq. (108) approximately as

$$A_v u_z^0 + A_v u_z^1 = c_d |u^0| u^0 + 2c_d |u^0| u^1.$$

Collecting terms of the same order (i.e. $O(1)$ and $O(\epsilon)$) one finds

$$\begin{aligned} A_v u_z^0 &= c_d |u^0| u^0, \\ A_v u_z^1 &= 2c_d |u^0| u^1. \end{aligned}$$

From this it follows that the instantaneous non-linear friction coefficient $2c_d |u^0|$ acting on the first order velocity field is in fact twice as high as the coefficient for the dominant zeroth order flow. Therefore, it is assumed that the linearised friction coefficient has the same dependency so that $s_{f_{0, M_0}} = s_{f_{0, M_4}} = 2s_{f_{0, M_2}}$, which indeed leads to an improvement of the calibration.

Based on minimizing the cost function, Eq. (107), for both the M_2 and M_4 tide, it follows that best results are obtained with $A_{v_0} = 0.061 \text{ m}^2 \text{ s}^{-1}$ and $s_{f_{0, M_2}} = 0.003 \text{ m s}^{-1}$. The corresponding values of the cost function are $f_{\text{cost}, M_2} = 0.21 \text{ m}$ and $f_{\text{cost}, M_4} = 0.09 \text{ m}$ for the M_2 and M_4 tide, respectively. In the next section, results will be presented based on the calibrated values for the constant turbulence parameters presented in this section.

6 Results

6.1 Water levels

6.1.1 Calibrated water levels

Figure 16 shows the longitudinal distribution of the amplitude and phase of the M_2 and M_4 water level that follows from the calibration as described in the previous section. For both constituents the phase is reasonably well reproduced by the model. For the amplitudes the quantitative agreement is less, but the qualitative features (in particular the amplification and damping of the M_2 tide) are recovered. Notably, at the end of the estuary, where the parameter $\epsilon = A_{M_2}/H$ is not necessarily much smaller than one, the model results for both the M_2 and M_4 amplitude deviate from the observations. For the M_4 tide, model results overestimate the amplitude by a factor of approximately two, which was also found with a similar model used by Schramkowski *et al.*, (2017).

6.1.2 Asymmetry of the vertical tide

At any location in the Scheldt estuary one can write the instantaneous water level $\zeta(x, t)$ as

$$\zeta(x, t) = A_{M_2}(x) \cos(\sigma t - \varphi_{M_2}(x)) + A_{M_4}(x) \cos(2\sigma t - \varphi_{M_4}(x)). \quad (109)$$

The shape of the water level curve is in general non-symmetric so that, for instance, the fall time t_{fall} elapsed between high water (HW) and low water (LW) is not equal to the rise time t_{rise} elapsed from LW to HW. This asymmetry can be expressed through the quantity δ_{asym} which is defined as the ratio of fall to rise time, i.e. (Friedrichs, 2010)

$$\delta_{\text{asym}} = \frac{t_{\text{fall}}}{t_{\text{rise}}}.$$

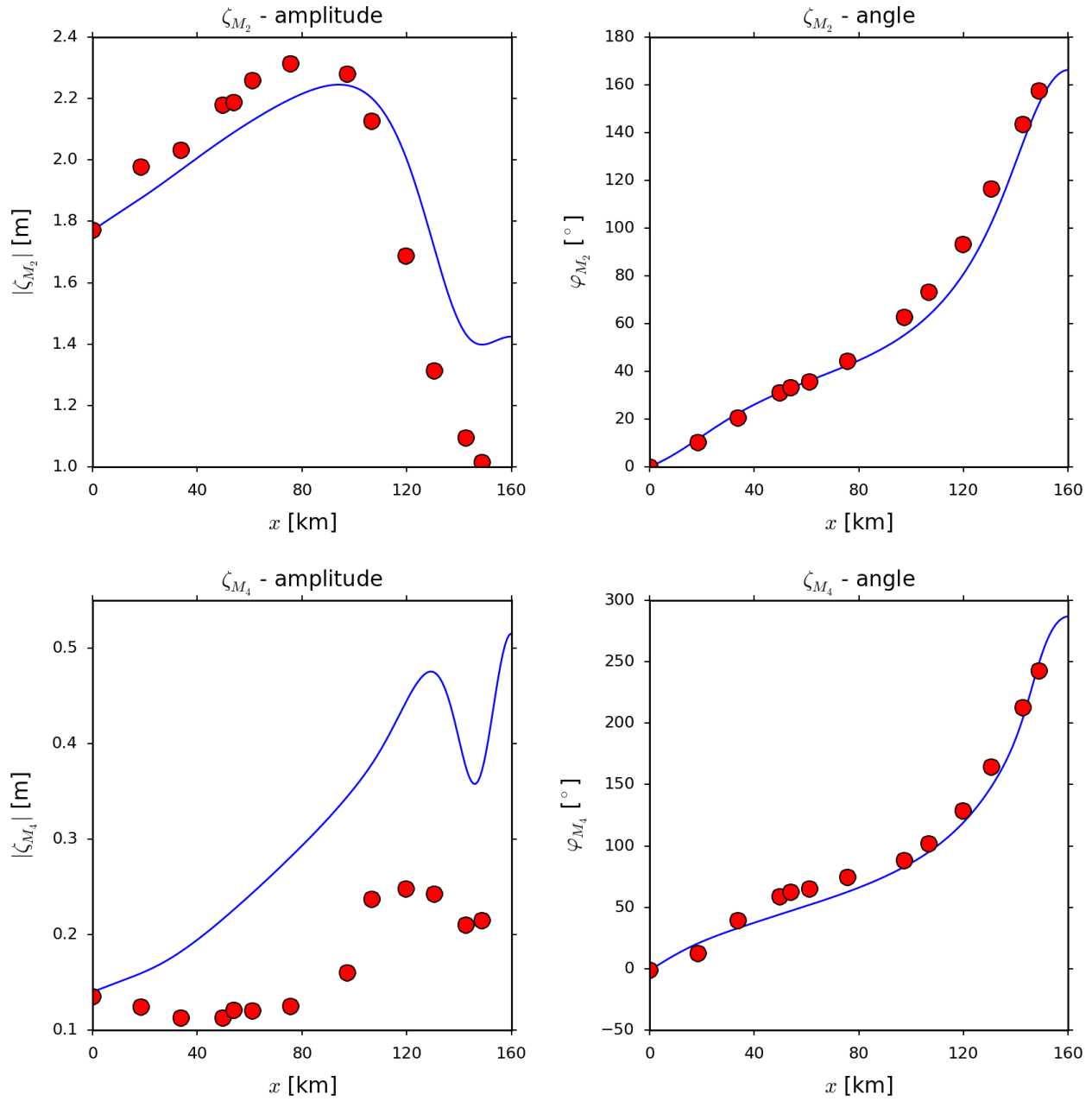
If $\delta_{\text{asym}} > 1$ the water level rises faster than it falls. On the other hand, if $\delta_{\text{asym}} < 1$ the fall of the tide lasts shorter than the rise. The asymmetry can be related to the phase difference $\Delta\varphi = \varphi_{M_4}(x) - 2\varphi_{M_2}(x)$. Indeed, from the assumption that M_4 tide is relatively weak compared to the M_2 contribution it follows that

$$\begin{aligned} t_{\text{fall}} &\approx \frac{P}{2} \left(1 - \frac{4}{\pi} \frac{A_{M_4}(x)}{A_{M_2}(x)} \sin \Delta\varphi \right), \\ t_{\text{rise}} &\approx \frac{P}{2} \left(1 + \frac{4}{\pi} \frac{A_{M_4}(x)}{A_{M_2}(x)} \sin \Delta\varphi \right), \end{aligned}$$

so that

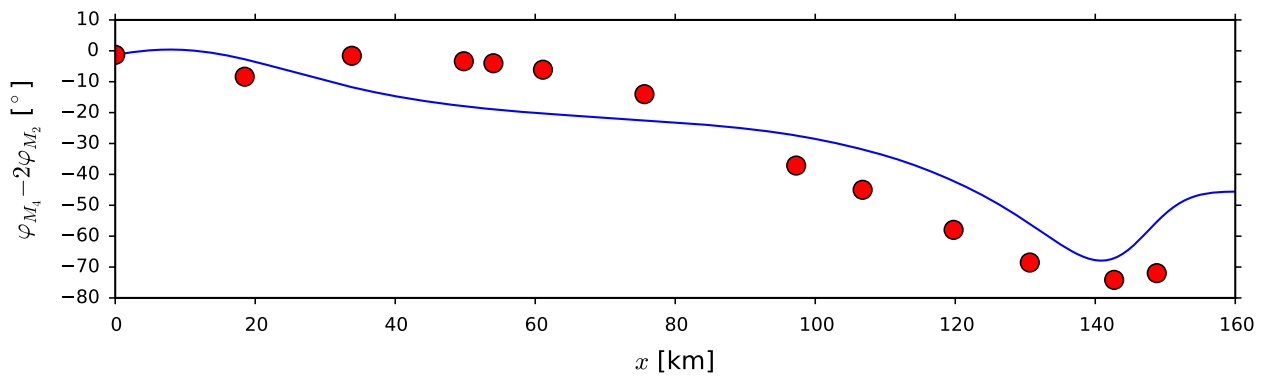
$$\delta_{\text{asym}} \approx 1 - \frac{8}{\pi} \frac{A_{M_4}(x)}{A_{M_2}(x)} \sin(\Delta\varphi).$$

Figure 17 shows the phase difference $\Delta\varphi$ from both model and measurements. While the data and model results show local deviations up to 20° , the phase difference is typically between -80° and zero degrees. From this it follows that the phase difference as shown in Fig. 17 actually corresponds to $\delta_{\text{asym}} > 1$ so that the vertical tide is rising faster than it falls. This agrees with the observed full tidal characteristics (i.e. with all constituents included, see Vandenbruwaene *et al.*, 2016a) so that the idealized model is able to reproduce observed qualitative asymmetry of the vertical tide.

Figure 16 – Amplitude (left panels) and phase (right panels) of the calibrated M_2 (top panels) and M_4 tide (bottom panels) as a function of the distance to Vlissingen.


Red dots denote measurements and blue lines correspond to the model results.

Figure 17 – Phase difference between M_2 and M_4 tide as a function of the distance to Vlissingen.



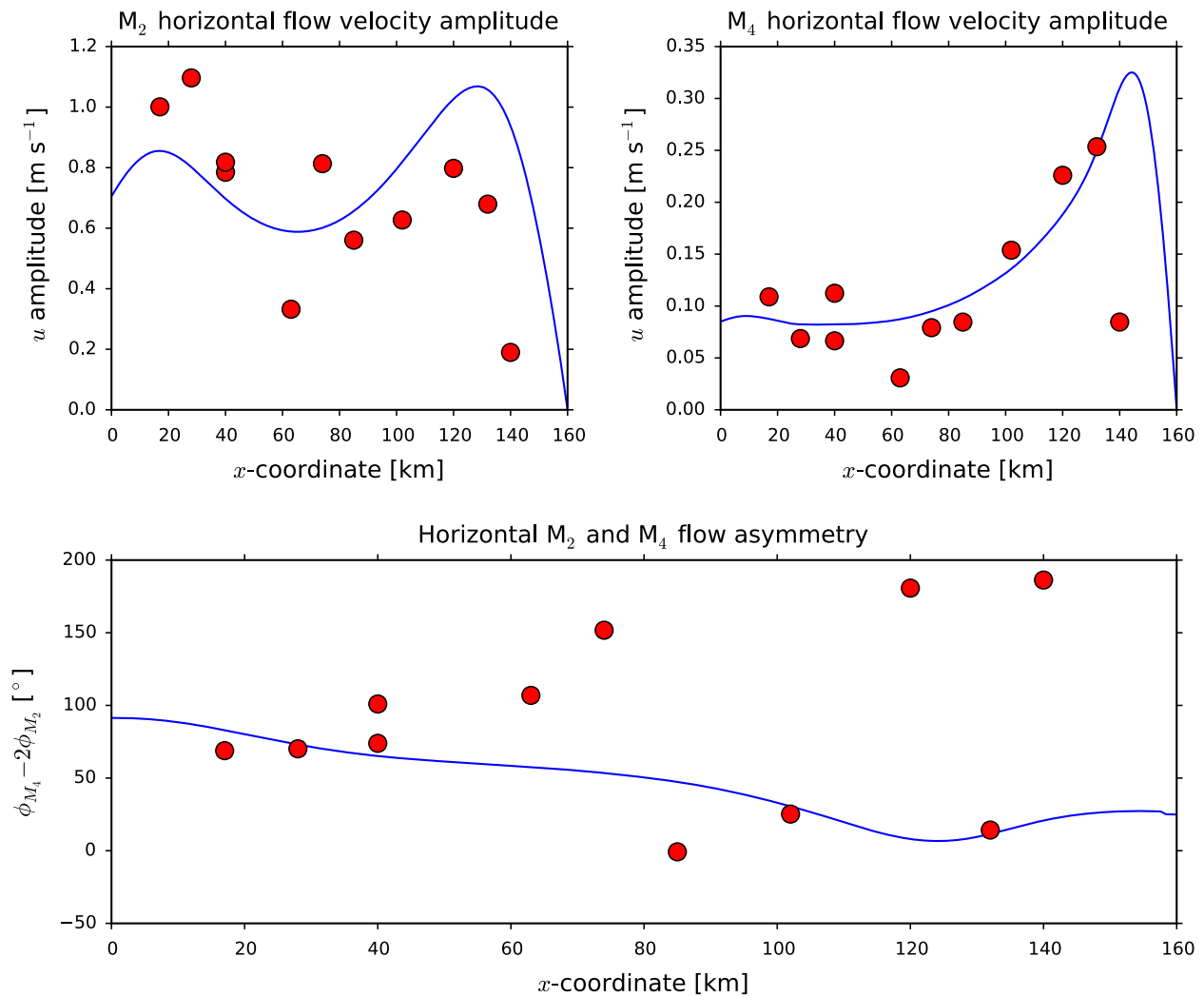
Red dots denote measurements and the blue line corresponds to the model results.

6.2 Tidal velocity

6.2.1 Amplitudes of the flow constituents

The top panels in Fig. 18 show a comparison between the amplitude of the computed and measured M_2 and M_4 flow velocity. While in both cases the order of magnitude roughly agrees, there is no accurate match. Note, however, that even though the vertical M_4 tidal amplitude is overestimated by a factor of approximately two (Fig. 16 middle left panel), the M_4 horizontal flow velocity amplitude shows remarkable agreement with the observations.

Figure 18 – Comparison between computed and observed tidal flow.



Top panels: amplitude of the M_2 (left) and M_4 (right) depth averaged velocity as a function of the distance to Vlissingen. The bottom panel shows the along-channel variation of the phase difference between M_2 and M_4 tidal velocity. Red dots denote measurements and blue lines correspond to the model results.

6.2.2 Asymmetry of the horizontal tide

Similar to Eq. (109) one can write the depth averaged tidal flow velocity as

$$\bar{u}(x, t) = \bar{u}_{M_2}(x) \cos(\sigma t - \phi_{M_2}(x)) + \bar{u}_{M_4}(x) \cos(2\sigma t - \phi_{M_4}(x)). \quad (110)$$

In general this flow is asymmetric which means, for instance, that ebb and flood flow velocities are of unequal magnitude. This has profound implications for the net transport of sediment and thus for, e.g., the dynamics of turbidity maxima and bed evolution (morphology). Horizontal tidal asymmetry is also one of the factors that plays a part in the regime shift hypothesis by Winterwerp and Wang, (2013)

To illustrate the effect of horizontal tidal asymmetry further we compute the net sediment flux by assuming a cubic relation between sediment flux q_{sed} and depth averaged velocity $\bar{u}$, i.e. $q_{\text{sed}} = p\bar{u}^3$. The net (i.e. tidally averaged) sediment flux $\langle q_{\text{sed}} \rangle$ according to Eq. (110) is given by

$$\langle q_{\text{sed}} \rangle = \frac{3}{4} p \bar{u}_{M_2}^2 \bar{u}_{M_4} \cos(\Delta\phi), \quad (111)$$

where $\Delta\phi = \phi_{M_4} - 2\phi_{M_2}$ is the phase difference between horizontal M_2 and M_4 tide. From this it follows that $\langle q_{\text{sed}} \rangle > 0$ (i.e. flood dominant flux) if $\Delta\phi$ is between -90° and 90° and $\langle q_{\text{sed}} \rangle < 0$ (ebb dominant flux) otherwise. Only for $\phi = -90^\circ$, or $\phi = 90^\circ$ one finds the net sediment flux to be zero.

In the bottom panel of Fig. 18 the along-channel variation of the tidal asymmetry $\Delta\phi$ is shown along with the values as obtained from the velocity measurements. It is seen that there is a reasonable agreement between model and data for the Western Scheldt part of the estuary ($x < 60$ km) where the tidal asymmetry angle lies along 90° , which implies a near-symmetric situation. In the Sea Scheldt the model gives flood dominant tidal asymmetry while the data show a large scatter with so that that here the comparison is inconclusive.

6.2.3 M_4 velocity component

Figure 19 shows the variation of the M_4 velocity amplitude and phase as a function of along-channel position x and height z . From these panels it is seen that the main contribution to the M_4 velocity in the Western Scheldt is due to the externally imposed M_4 tide (middle panel left). On the other hand, the Stokes flow (bottom panel left) and the flow associated with the stress free surface condition (bottom panel right) are the main contributors to the M_4 velocity in the Upper Sea Scheldt.

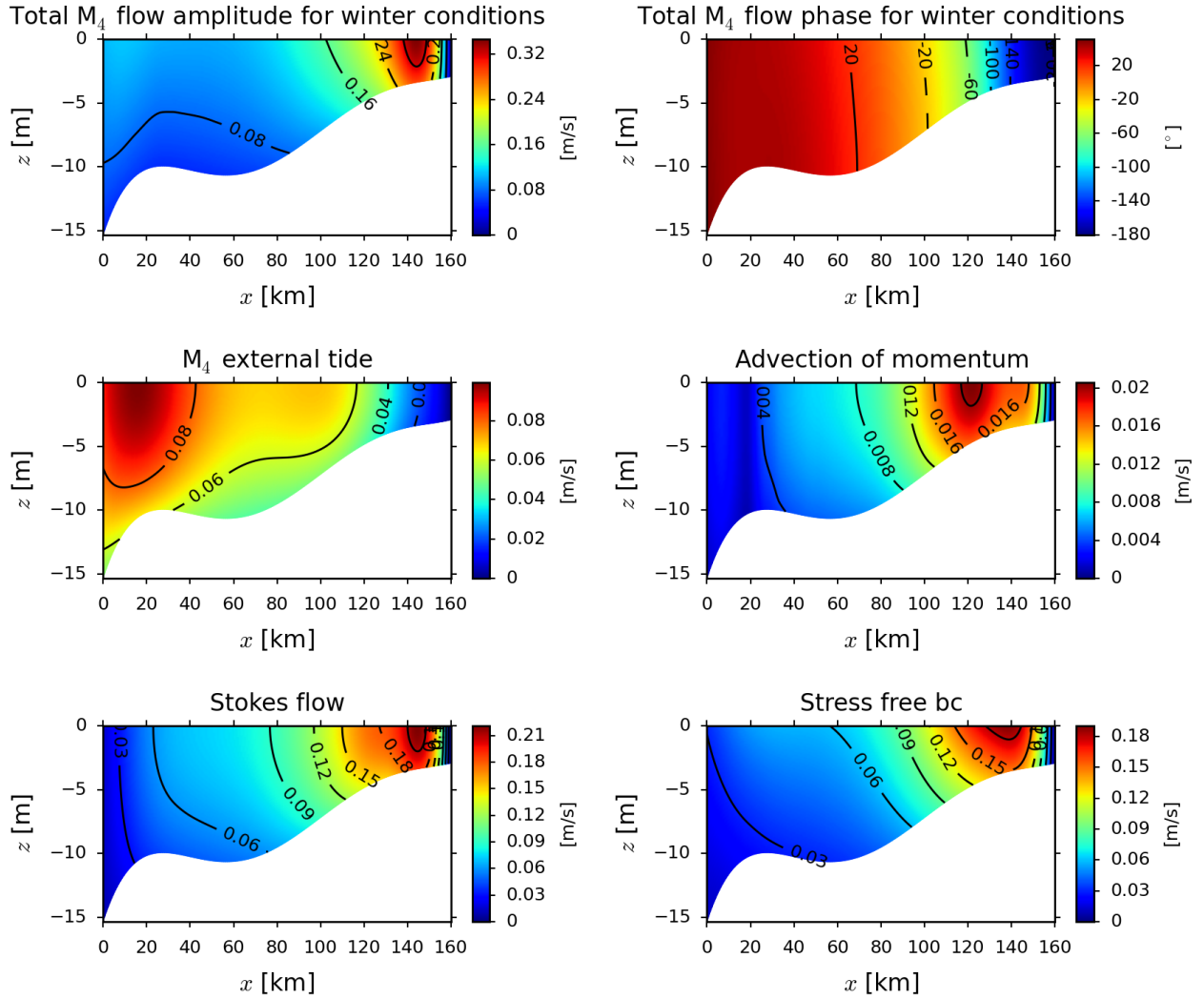
6.3 Residual flow

Below, the residual flow patterns for both winter and summer conditions will be discussed.

6.3.1 Winter conditions

River flow Figure 20 shows both the total residual flow and the individual flow components for the 2009 winter conditions (i.e. $Q = 110 \text{ m}^3 \text{ s}^{-1}$). The total flow is directed outward everywhere, with maximum velocities $\sim -0.9 \text{ m s}^{-1}$ occurring near the weir. It follows that the flow near the landward boundary is predominantly a result of the river flow (top right panel).

Stokes return flow The next important contribution is due to the Stokes return flow (middle panel on the right). It is always negative and for the calibrated model it can reach values up to -0.14 m s^{-1} . This flow is generated by a nonzero correlation between the water level and surface velocity (i.e. the quantity γ , see Sect.

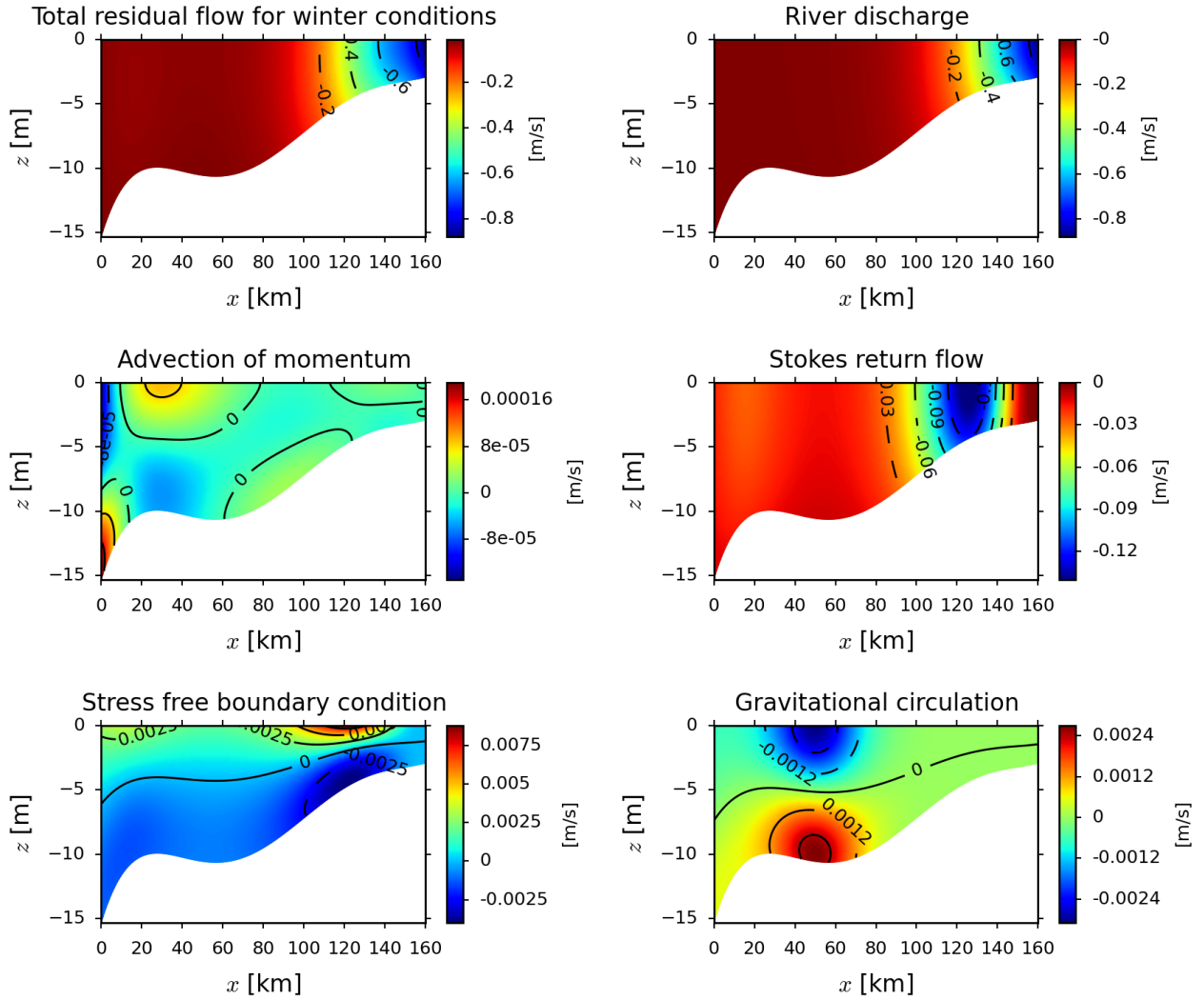
Figure 19 – M_4 amplitude (top left) and phase (top right) as function of position inside the Scheldt basin.


The other four panels give the individual contributions of external M_4 tide (middle left), acceleration (middle right), Stokes flow (bottom left) and no-stress boundary condition (bottom right).

3.2.2). Physically, this correlation expresses the fact that (on average) more water is flowing landward during high water than seaward during low water. Hence, the tide induces a net landward transport, which would lead to an unbounded accumulation of water near the weir. This is counteracted by a net return flow, which is the flow denoted by u^{stokes} . The strength of u^{stokes} scales linearly with γ , which stems from the product of M_2 water level and surface velocity. These quantities can be expressed as

$$\begin{aligned}\zeta &= \mathcal{A} \cos(\sigma t - \varphi_{M_2}), \\ u &= \mathcal{U} \cos(\sigma t - \psi_{M_2}),\end{aligned}$$

so that $\gamma = \frac{1}{2} \mathcal{A} \mathcal{U} \cos(\psi_{M_2} - \varphi_{M_2})$. Hence, the Stokes return flow depends on the phase difference between surface velocity and water level. This quantity is shown in Fig. 21. Clearly, $\psi_{M_2} - \varphi_{M_2}$ is 90° at the weir, which is due to the local standing wave character of the tidal wave since both landward propagating and reflected wave have nearly equal amplitude. Furthermore, it is seen that the phase difference has a minimum $\sim 55^\circ$ around $x = 125$ km which implies a relatively strong correlation with γ . At this location also a local maximum of Stokes flow strength is observed in the model, so it appears that this maximum return flow is related to the relatively low value of the phase difference between water level and surface velocity.

Figure 20 – Calculated residual along-channel flow velocities (contours/colors) for winter conditions based on data from the Scheldt Estuary as a function of channel depth z and distance to Vlissingen x .


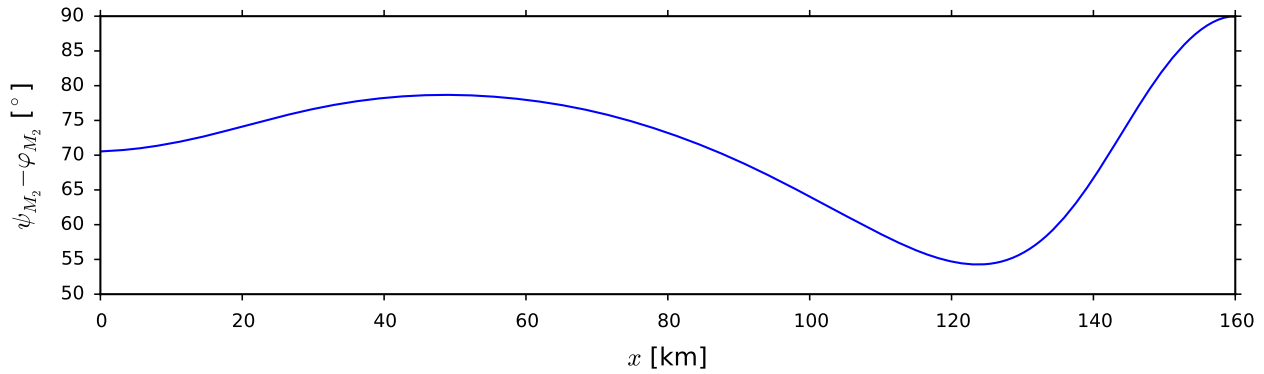
The top left panel depicts the total residual flow, and the other panels depict a contribution to this total residual flow in isolation, i.e. the contribution of the river discharge (top right panel), the advection of momentum (middle left panel), the Stokes flow (middle right panel), the stress free boundary condition (bottom left panel) and the gravitational circulation (bottom right panel).

Gravitational circulation A remarkable result is that the gravitational circulation is fairly weak (Fig. 20, bottom right panel), yielding maximum flow velocities up to 2.5 mm s^{-1} . Note that the maximum flow velocities occurs around $x = 50 \text{ km}$, which is to be expected since gravitational circulation scales with the local salinity gradient which has a maximum around $x = x_c = 48 \text{ km}$ (see Sect. 4.5).

Flow induced by the no-stress surface condition The residual flow that results because of the no-stress boundary condition at the mean water level $z = 0$ is shown in the bottom left panel of Fig. 20. Like gravitational circulation it is fairly weak (maximum flow velocities well below 1 cm s^{-1}). The along-channel location of the flow maximum can be understood by considering the corresponding forcing, namely the quantity $\chi(x, t)$ (Sect. 3.2.2). Closer inspection of this term reveals that it's along-channel variation is governed by $\zeta_0 \partial \zeta_0 / \partial x$. Assuming that the vertical M_2 tide is a predominantly propagating wave, it's functional form can be written as

$$\zeta_0 = \mathcal{A}(x) \cos(\sigma t - \varphi_{M_2}(x)) ,$$

Figure 21 – Phase difference $\psi_{M_2} - \varphi_{M_2}$ between the horizontal M_2 tide or horizontal M_2 flow velocity and the vertical M_2 tide as a function of the distance to Vlissingen.



so that

$$\left\langle \zeta_0 \frac{\partial \zeta_0}{\partial x} \right\rangle = \frac{1}{2} \mathcal{A} \frac{d\mathcal{A}}{dx} = \frac{1}{4} \frac{d\mathcal{A}^2}{dx}.$$

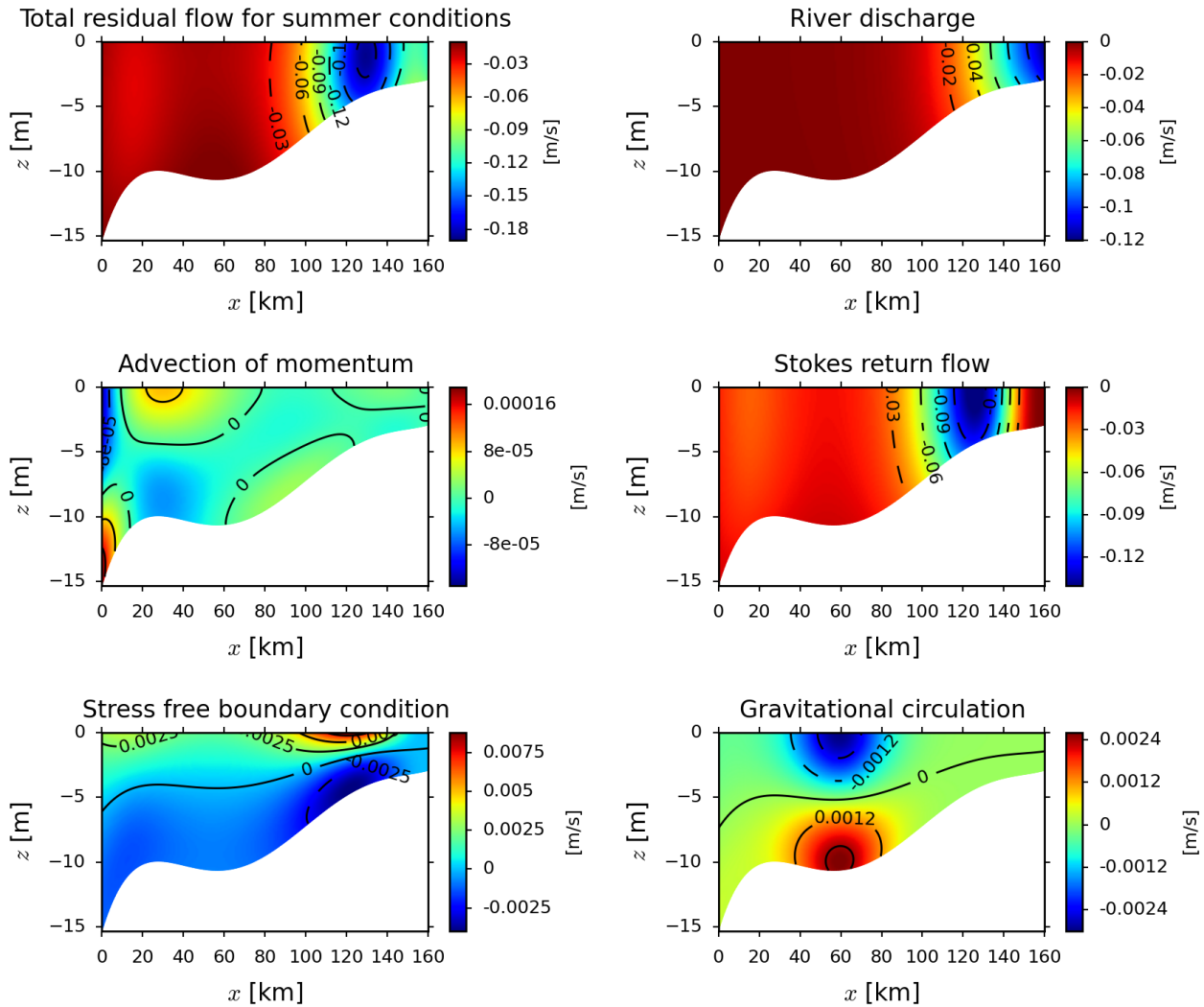
Hence the maximum flow velocity will occur where the M_2 tidal amplitude has a maximum gradient. From the top left panel of Fig. 16 it is seen that this maximum gradient occurs around $x = 120$ km, where the tidal wave is strongly damped. This is also the position where the maximum no-stress residual flow component is found.

6.3.2 Summer conditions

Figure 22 shows the residual flow patterns for summer conditions ($Q = 15 \text{ m}^3 \text{ s}^{-1}$). Note that the discharge only affects the river flow component and gravitational circulation, as the other contributions are forced by the M_2 tide which is unaffected by the river flow. Compared to winter conditions the total residual flow is much weaker now (maximum velocities $\sim -0.19 \text{ m s}^{-1}$), and the river flow contribution is much smaller ($\sim -0.12 \text{ m s}^{-1}$) which reflects the much lower river discharge. In fact, the Stokes return flow is the most important contributor to the total residual transport with maximum flow velocities up to -0.14 m s^{-1} (Fig. 22, middle panel right). Hence, the relative importance of the Stokes return flow is now approximately twice that of the river flow.

The strength of gravitational circulation has not altered compared to the winter situation, although the location of the maximum has now shifted landward to $x = 60$ km. This position agrees with the location of the maximum salinity gradient $ds(x)/dx$ (Sect. 4.5).

Figure 22 – Calculated residual flow velocities (contours/colors) for summer conditions based on data from the Scheldt Estuary as a function of channel depth z and distance to Vlissingen x .



The top panel depicts the total residual flow, the middle panel depicts the contribution of the river discharge and the bottom panel depicts the contribution of the gravitational circulation.

7 Conclusions

From the results of the calibrated model the following conclusions are drawn,

1. The model gives a reasonable quantitative agreement with the observed M_2 and M_4 phase of the vertical tide throughout the Scheldt estuary.
2. The model gives a good qualitative agreement with the observed amplitude variation of the M_2 water level along the basin.
3. The magnitude of the calculated M_4 tidal amplitude is overestimated by a factor of approximately two compared to measurements.
4. The amplitudes of the horizontal M_2 and M_4 tide (velocity) are of similar order of magnitude as the values obtained from measurements. However, the spatial (i.e. along-estuary) distribution shows no clear agreement. Furthermore, even though the vertical M_4 tidal amplitude is overestimated by a factor two, the horizontal M_4 tidal amplitude shows remarkable agreement with observations.
5. In the Western Scheldt the asymmetry of the horizontal tide is in reasonable agreement with data. For the Sea Scheldt the model predicts a flood dominant situation while the measurements show a large scatter that allows no conclusion regarding the tidal asymmetry.
6. The M_4 flow in the Western Scheldt is mainly driven by the external M_4 tide, while the Stokes flow and the flow due to the stress free boundary condition are the primary contributors of the M_4 velocity in the Upper Sea Scheldt.
7. For winter conditions (high discharge) the residual flow is predominantly a result of river runoff.
8. For summer conditions (low discharge) the Stokes return flow gives the largest contribution to the residual flow pattern, with river flow being a factor two smaller.
9. Gravitational circulation does not give an important contribution to the total residual flow .

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A1 Reduction and solution of the model equations

This appendix is an extension of Section (3.2). It gives a detailed analysis according to the perturbation method introduced in that section. After deriving the dimensionless model equations and subsequent ordering according to the relative importance of the various terms, this appendix gives the dimensionless analytical solutions to the model equations. Notice that this is different than the solutions given in the main text, which are dimensional. The reason to do so is that the dimensionless equations are useful to identify specific important system parameters, whereas the dimensional equations and solutions reach a more general public.

A1.1 Perturbation analysis

For clarity of the analysis, we repeat the model equations (11)-(20)

$$u_t + uu_x + wu_z = -g\zeta_x + g\beta_s \langle s_x \rangle (z - \zeta) + A_v u_{zz}, \quad (112)$$

$$u_x + w_z + \frac{B_x}{B} u = 0, \quad (113)$$

$$\zeta_t + \left(\frac{\partial}{\partial x} + \frac{B_x}{B} \right) \int_{-H}^{\zeta} u dz = 0, \quad (114)$$

and boundary conditions

$$\zeta = A_{M_2} \cos(\sigma t) + A_{M_4} \cos(2\sigma t - \varphi), \quad \text{at } x = 0, \quad (115)$$

$$\int_{-H}^{\zeta} u dz = -\frac{Q}{B}, \quad \text{at } x = L, \quad (116)$$

$$w = \zeta_t + u\zeta_x, \quad \text{at } z = \zeta, \quad (117)$$

$$A_v u_z = 0, \quad \text{at } z = \zeta, \quad (118)$$

$$A_v u_z = s_f u, \quad \text{at } z = -H, \quad (119)$$

$$w = -uH_x, \quad \text{at } z = -H. \quad (120)$$

For the perturbation analysis, first the model equations and boundary conditions are scaled according to typical variable scales denoted in Table 9. Dimensionless variables are denoted with a tilde, $(\tilde{\cdot})$. The typical scales for the horizontal and vertical velocity are derived scales that follow from the depth-integrated and regular continuity equations (see below for details).

Table 9 – Typical variable scales.

Physical quantity	Typical scale	Symbol	Variable
Time	M ₂ tidal frequency	σ	$t = \sigma^{-1}\tilde{t}$
Surface elevation	M ₂ tidal amplitude	A_{M_2}	$\zeta = A_{M_2}\tilde{\zeta}$
Horizontal coordinate	Estuary length	L	$x = L\tilde{x}$
Vertical coordinate	Average estuary depth at entrance	H_0	$z = H_0\tilde{z}$
Local water depth	Average estuary depth at entrance	H_0	$H = H_0\tilde{H}$
Local channel width	Average estuary width at entrance	B_0	$B = B_0\tilde{B}$
Salinity gradient	Typical salinity gradient	$\mathcal{S}_x$	$s_x = \mathcal{S}_x\tilde{s}_x$
Physical quantity	Derived typical scale	Symbol	Variable
Horizontal velocity	Typical horizontal velocity of the M ₂ tide	$U = \frac{\sigma A_{M_2} L}{H_0}$	$u = U\tilde{u}$
Vertical velocity	Typical vertical velocity of the M ₂ tide	$W = \frac{H_0 U}{L} = \sigma A_{M_2}$	$w = W\tilde{w}$

A1.1.1 Scaling the model equations

Substituting the scaled variables from Table 9 into the depth-integrated continuity equation (114) and writing $\int_{-H}^{\zeta} u dz = H\bar{u}$, where the bar denotes the depth-integrated quantity, yields

$$\sigma A_{M_2} \tilde{\zeta}_{\tilde{t}} + \frac{H_0 U}{L} (\tilde{H} \tilde{u})_{\tilde{x}} + \frac{H_0 U}{L} \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \tilde{H} \tilde{u} = 0. \quad (121)$$

Thus, it follows that an appropriate scale U for the velocity is

$$U = \frac{\sigma A_{M_2} L}{H_0}. \quad (122)$$

Similarly, the derived vertical velocity scale W follows from substituting the scaled variables into Eq. (113)

$$\frac{U}{L} \tilde{u}_{\tilde{x}} + \frac{W}{H_0} \tilde{w}_{\tilde{z}} + \frac{U}{L} \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \tilde{u} = 0, \quad (123)$$

and thus

$$W = \frac{H_0 U}{L} \cdot \max \left(1, \frac{B_x}{B} \right). \quad (124)$$

The dimensionless momentum equation is obtained by substituting the scaled variables into Eq. (112)

$$\sigma U \tilde{u}_{\tilde{t}} + \frac{U^2}{L} \tilde{u} \tilde{u}_{\tilde{x}} + \frac{W U}{H_0} \tilde{w} \tilde{u}_{\tilde{z}} = -\frac{g A_{M_2}}{L} \tilde{\zeta}_{\tilde{x}} + g \beta_s \mathcal{S}_x \langle \tilde{s}_{\tilde{x}} \rangle (H_0 \tilde{z} - A_{M_2} \tilde{\zeta}) + \frac{U}{H_0^2} A_v \tilde{u}_{\tilde{z}\tilde{z}}. \quad (125)$$

Rewriting leads to

$$\tilde{u}_{\tilde{t}} + \frac{A_{M_2}}{H_0} [\tilde{u} \tilde{u}_{\tilde{x}} + \tilde{w} \tilde{u}_{\tilde{z}}] = -\frac{1}{\lambda^2} \tilde{\zeta}_{\tilde{x}} + \mu \langle \tilde{s}_{\tilde{x}} \rangle (\tilde{z} - \frac{A_{M_2}}{H_0} \tilde{\zeta}) + \frac{1}{\sigma H_0^2} A_v \tilde{u}_{\tilde{z}\tilde{z}}, \quad (126)$$

where

$$\lambda = \frac{L}{L_w} = \frac{\sigma L}{\sqrt{gH_0}}, \quad (127)$$

and

$$\mu = \frac{gH_0}{U} \frac{\beta_s \mathcal{S}_x}{\sigma}. \quad (128)$$

Here, λ is the ratio of the estuary length L and the frictionless tidal wave length L_w up to a factor 2π . Furthermore, μ is a factor determining the magnitude of the salinity gradient.

The ratio between the tidal wave amplitude and the average entrance depth, A_{M_2}/H_0 , that multiplies the advection term on the left-hand side in Eq. (126) is the barotropic nonlinearity parameter (Stokes, 1847), and denoted by

$$\epsilon = \frac{A_{M_2}}{H_0}. \quad (129)$$

Although ϵ may approach unity in very shallow systems, it is small but finite in most systems and is thus appropriate as a perturbation expansion parameter.

The other factors that appear in the dimensionless momentum equation (126) can be related to the magnitude of ϵ . Firstly, λ is approximately 1 when the estuary length is of the same order as the frictionless tidal wave length. For most estuaries including the Schelde ($\lambda = 1.8$) this is a good approximation. Secondly, it is assumed that the factor μ in front of the salinity gradient is of order ϵ . From observations in well-mixed estuaries it usually follows that currents driven by the salinity gradient are small compared to tidally driven currents. Thirdly, a typical scale for the eddy viscosity follows from the stationary barotropic momentum balance $A_v u_{zz} = g\zeta_x$. Assuming that the frictional boundary layer thickness is approximately equal to the water depth (good approximation for the estuaries that are of interest in the project, i.e. Schelde and Ems estuaries) and A_v is independent of z , it follows that $A_v \sim \sigma H_0^2$ and the factor in front of $\tilde{u}_{zz}$ becomes 1.

Finally, substituting the scaled variables from Table 9 and the small parameter ϵ into Eqs. (112)-(114) and rearranging terms leads to the following dimensionless model equations

$$\tilde{u}_{\tilde{t}} + \epsilon [\tilde{u}\tilde{u}_{\tilde{x}} + \tilde{w}\tilde{u}_{\tilde{z}}] = -\frac{1}{\lambda^2} \tilde{\zeta}_{\tilde{x}} + \mu \langle \tilde{s}_{\tilde{x}} \rangle (\tilde{z} - \epsilon \tilde{\zeta}) + \delta \tilde{u}_{\tilde{z}\tilde{z}}, \quad (130)$$

$$\tilde{u}_{\tilde{x}} + \tilde{w}_{\tilde{z}} + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \tilde{u} = 0, \quad (131)$$

$$\tilde{\zeta}_{\tilde{t}} + \left(\frac{\partial}{\partial \tilde{x}} + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \right) \int_{-\tilde{H}}^{\epsilon \tilde{\zeta}} \tilde{u} d\tilde{z} = 0, \quad (132)$$

where $\delta = A_v/(\sigma H_0^2)$ is the squared ratio of the thickness of the Stokes boundary layer and the water depth.

A1.1.2 Scaling the boundary conditions

Similar to the model equations, the boundary conditions Eq. (115)-(120) are scaled according to the scaled variables from Table 9 and the small parameter ϵ . This results in the following dimensionless boundary conditions

$$\tilde{\zeta} = \cos \tilde{t} + \frac{A_{M_4}}{A_{M_2}} \cos(\tilde{t} - \varphi), \quad \text{at } \tilde{x} = 0, \quad (133)$$

$$\int_{-\tilde{H}}^{\epsilon \tilde{\zeta}} \tilde{u} d\tilde{z} = -\frac{Q}{B_0 H_0 U \tilde{B}}, \quad \text{at } \tilde{x} = 1, \quad (134)$$

$$\tilde{w} = \tilde{\zeta}_{\tilde{t}} + \epsilon \tilde{u} \tilde{\zeta}_{\tilde{x}}, \quad \text{at } \tilde{z} = \epsilon \tilde{\zeta}, \quad (135)$$

$$A_v \tilde{u}_{\tilde{z}} = 0, \quad \text{at } \tilde{z} = \epsilon \tilde{\zeta}, \quad (136)$$

$$\tilde{u}_{\tilde{z}} = \theta \tilde{u}, \quad \text{at } \tilde{z} = -\tilde{H}, \quad (137)$$

$$\tilde{w} = -\tilde{u} \tilde{H}_{\tilde{x}}, \quad \text{at } \tilde{z} = -\tilde{H}, \quad (138)$$

where $\theta = s_f H_0 / A_v$ is the dimensionless slip parameter. If $\theta \ll 1$ then the bottom is stress free. If on the other hand $\theta \gg 1$ the velocity at the bed vanishes and the boundary condition reduces to the no-slip condition.

A1.1.3 Perturbation technique

The final step before obtaining the ordered system of equations, is to approximate the solution of the dimensionless Eqs. (130)-(132) together with the appropriate boundary conditions Eqs. (133)-(138) for $\epsilon \ll 1$. Hereby, it is assumed that the following parameters are related to ϵ as followed: $A_{M_4}/A_{M_2} = \mathcal{O}(\epsilon)$, $\lambda = \mathcal{O}(1)$, $\mu = \mathcal{O}(\epsilon)$, $\delta = \mathcal{O}(1)$, $B_x/B = \mathcal{O}(1)$ and $\theta = \mathcal{O}(1)$. Expanding the physical variables $\tilde{u}$, $\tilde{w}$ and $\tilde{\zeta}$ in power series of the small parameter ϵ results in

$$\tilde{u} = \tilde{u}^0 + \tilde{u}^1 + \tilde{u}^2 + \dots \quad (139)$$

$$\tilde{w} = \tilde{w}^0 + \tilde{w}^1 + \tilde{w}^2 + \dots \quad (140)$$

$$\tilde{\zeta} = \tilde{\zeta}^0 + \tilde{\zeta}^1 + \tilde{\zeta}^2 + \dots \quad (141)$$

Here, superscripts denote the order of ϵ . Substituting these expansions in the dimensionless model equations and boundary conditions and collecting terms of equal powers of ϵ , results in the system of equations at different orders. In this study, we are interested in the dynamics of the system at leading order, $\mathcal{O}(1)$, and first order, $\mathcal{O}(\epsilon^1)$.

The leading order system is given by

$$\delta \tilde{u}_{\tilde{z}\tilde{z}}^0 - \tilde{u}_t^0 = \frac{1}{\lambda^2} \tilde{\zeta}_x^0, \quad (142)$$

$$\tilde{u}_x^0 + \tilde{w}_{\tilde{z}}^0 + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \tilde{u}^0 = 0, \quad (143)$$

$$\tilde{\zeta}_t^0 + \left(\frac{\partial}{\partial \tilde{x}} + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \right) \int_{-\tilde{H}}^0 \tilde{u}^0 d\tilde{z} = 0, \quad (144)$$

with boundary conditions

$$\tilde{\zeta}^0 = \cos \tilde{t}, \quad \text{at } \tilde{x} = 0, \quad (145)$$

$$\int_{-\tilde{H}}^0 \tilde{u}^0 d\tilde{z} = 0, \quad \text{at } \tilde{x} = 1, \quad (146)$$

$$\tilde{w}^0 = \tilde{\zeta}_t^0, \quad \text{at } \tilde{z} = 0, \quad (147)$$

$$A_v \tilde{u}_{\tilde{z}}^0 = 0, \quad \text{at } \tilde{z} = 0, \quad (148)$$

$$\tilde{u}_{\tilde{z}}^0 = \theta \tilde{u}^0, \quad \text{at } \tilde{z} = -\tilde{H}, \quad (149)$$

$$\tilde{w}^0 = -\tilde{u}^0 \tilde{H}_{\tilde{x}}, \quad \text{at } \tilde{z} = -\tilde{H}. \quad (150)$$

The first order system is given by

$$\delta \tilde{u}_{\tilde{z}\tilde{z}}^1 - \tilde{u}_t^1 = \frac{1}{\lambda^2} \tilde{\zeta}_x^1 + \tilde{u}^0 \tilde{u}_x^0 + \tilde{w}^0 \tilde{u}_{\tilde{z}}^0 - \mu \langle \tilde{s}_x \rangle \tilde{z}, \quad (151)$$

$$\tilde{u}_x^1 + \tilde{w}_{\tilde{z}}^1 + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \tilde{u}^1 = 0, \quad (152)$$

$$\tilde{\zeta}_t^1 + \left(\frac{\partial}{\partial \tilde{x}} + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \right) \left(\int_{-\tilde{H}}^0 \tilde{u}^1 d\tilde{z} + \tilde{\zeta}^0 \tilde{u}^0 \Big|_{\tilde{z}=0} \right) = 0, \quad (153)$$

with boundary conditions

$$\tilde{\zeta}^1 = \frac{A_{M_4}}{A_{M_2}} \cos(2\tilde{t} - \varphi), \quad \text{at } \tilde{x} = 0, \quad (154)$$

$$\int_{-\tilde{H}}^0 \tilde{u}^1 d\tilde{z} + \tilde{\zeta}^0 \tilde{u}^0 \Big|_{\tilde{z}=0} = -\frac{Q}{B_0 H_0 U \tilde{B}}, \quad \text{at } \tilde{x} = 1, \quad (155)$$

$$\tilde{w}^1 = \tilde{\zeta}_t^1 + \tilde{u}^0 \tilde{\zeta}_x^0 - \tilde{\zeta}^0 \tilde{w}_{\tilde{z}}^0, \quad \text{at } \tilde{z} = 0, \quad (156)$$

$$A_v \tilde{u}_{\tilde{z}}^1 = -A_v \tilde{\zeta}^0 \tilde{u}_{\tilde{z}\tilde{z}}^0, \quad \text{at } \tilde{z} = 0, \quad (157)$$

$$\tilde{u}_{\tilde{z}}^1 = \theta \tilde{u}^1, \quad \text{at } \tilde{z} = -\tilde{H}, \quad (158)$$

$$\tilde{w}^1 = -\tilde{u}^1 \tilde{H}_{\tilde{x}}, \quad \text{at } \tilde{z} = -\tilde{H}. \quad (159)$$

In deriving the ordered depth-integrated expressions appearing in the depth-integrated continuity equation (132) and the boundary condition at the riverine side (134), use is made of a Taylor series expansion as exemplified below. Starting with a nondimensional integral form (in this case the integral on the right-hand side of Eq. (132)) and using the expansions of the physical variables leads to the following expression for the depth-integrated velocity I ,

$$I = \int_{-\tilde{H}}^{\epsilon\tilde{\zeta}^0 + \epsilon^2\tilde{\zeta}^1} (\tilde{u}^0 + \epsilon\tilde{u}^1) d\tilde{z}. \quad (160)$$

Introducing

$$F^0(\eta) = \int_{-\tilde{H}}^{\eta} \tilde{u}^0 d\tilde{z} \quad \text{and} \quad F^1(\eta) = \int_{-\tilde{H}}^{\eta} \tilde{u}^1 d\tilde{z}, \quad (161)$$

where $\eta = \epsilon\tilde{\zeta}^0 + \epsilon^2\tilde{\zeta}^1$, Eq. (160) can now be rewritten as

$$I = F^0(\eta) + \epsilon F^1(\eta). \quad (162)$$

Using a Taylor series expansion results in

$$I = F^0(0) + \eta \frac{\partial F^0(0)}{\partial z} + \epsilon F^1(0) + \epsilon \eta \frac{\partial F^1(0)}{\partial z} + \text{h.o.t.} \quad (163)$$

where h.o.t. denote higher order terms. Resubstituting the expressions for F^0 and F^1 from Eq. (162) leads to

$$I = \int_{-\tilde{H}}^0 \tilde{u}^0 d\tilde{z} + (\epsilon\tilde{\zeta}^0 + \epsilon^2\tilde{\zeta}^1) \tilde{u}^0|_{\tilde{z}=0} + \epsilon \int_{-\tilde{H}}^0 \tilde{u}^1 d\tilde{z} + (\epsilon^2\tilde{\zeta}^0 + \epsilon^3\tilde{\zeta}^1) \tilde{u}^1|_{\tilde{z}=0}. \quad (164)$$

Collecting terms up to $\mathcal{O}(\epsilon^1)$ finally results in

$$I = \int_{-\tilde{H}}^0 \tilde{u}^0 d\tilde{z} + \epsilon \left[\int_{-\tilde{H}}^0 \tilde{u}^1 d\tilde{z} + \tilde{\zeta}^0 \tilde{u}^0|_{\tilde{z}=0} \right]. \quad (165)$$

A1.2 Solutions

A1.2.1 Leading order solutions

Since the leading order equations are forced by a single M_2 tidal component and the differential equations do not contain time-dependent coefficients, solutions are of the following form

$$(\tilde{u}^0, \tilde{w}^0, \tilde{\zeta}^0) = \Re \left\{ \left(\hat{\tilde{u}}^0(\tilde{x}, \tilde{z}), \hat{\tilde{w}}^0(\tilde{x}, \tilde{z}), \hat{\tilde{\zeta}}^0(\tilde{x}, \tilde{z}) \right) e^{i\tilde{t}} \right\}, \quad (166)$$

where $\Re\{\cdot\}$ denotes the real part of the expression between braces and i is the imaginary unit. In Eq. (166), $\hat{u}^0$, $\hat{w}^0$, $\hat{\zeta}^0$ are the dimensionless complex amplitudes of the horizontal velocity, the vertical velocity and the surface elevation, respectively.

Substituting the trial solution Eq. (166) into the momentum and continuity equations, Eqs. (142)-(144), leads to

$$\delta \hat{u}_{\tilde{z}\tilde{z}}^0 - i \hat{u}^0 = \frac{1}{\lambda^2} \hat{\zeta}_{\tilde{x}}^0, \quad (167)$$

$$\hat{u}_{\tilde{x}}^0 + \hat{w}_{\tilde{z}}^0 + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \hat{u}^0 = 0, \quad (168)$$

$$i \hat{\zeta}^0 + \left(\frac{\partial}{\partial \tilde{x}} + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \right) \int_{-\tilde{H}}^0 \hat{u}^0 d\tilde{z} = 0, \quad (169)$$

Horizontal velocity

First, the momentum equation (167) is solved to obtain an expression for the horizontal velocity amplitude $\hat{u}^0$. This equation is subject to the no stress boundary condition (148) at the free surface $\tilde{z} = 0$ and the partial slip boundary condition (149) at the bottom $\tilde{z} = -\tilde{H}$. Using the solution (166), these boundary conditions transform into

$$A_v \hat{u}_{\tilde{z}}^0(\tilde{x}, 0) = 0, \quad (170)$$

$$\hat{u}_{\tilde{z}}^0(\tilde{x}, -\tilde{H}) - \theta \hat{u}^0(\tilde{x}, -\tilde{H}) = 0, \quad (171)$$

respectively. The general solution of Eq. (167) can be expressed in the form

$$\hat{u}^0 = \tilde{u}_c^0 + \tilde{u}_p^0, \quad (172)$$

where the particular solution $\tilde{u}_p^0$ is any specific function that satisfies the nonhomogeneous equation (167) and the complementary solution $\tilde{u}_c^0$ is a general solution of the corresponding homogeneous equation. As it is assumed here that the vertical eddy viscosity is independent of $\tilde{z}$, the homogeneous equation can be written as

$$\hat{u}_{\tilde{z}\tilde{z}}^0 - \frac{i}{\delta} \hat{u}^0 = 0, \quad (173)$$

and the characteristic equation is

$$\tilde{r}_{M_2}^2 - \frac{i}{\delta} = 0. \quad (174)$$

Consequently, the complementary solution $\tilde{u}_c^0$ reads

$$\tilde{u}_c^0 = C_1 e^{\tilde{r}_{M_2} \tilde{z}} + C_2 e^{-\tilde{r}_{M_2} \tilde{z}}, \quad (175)$$

with

$$\tilde{r}_{M_2} = \sqrt{\frac{i}{\delta}}. \quad (176)$$

From Eq. (167) it follows that $\tilde{u}_p^0$ is of the form $C_3 \hat{\zeta}_{\tilde{x}}^0$ and, thus, $\tilde{u}_{p,\tilde{z}}^0 = \tilde{u}_{p,\tilde{z}\tilde{z}}^0 = 0$. Substitution in Eq. (167) leads to

$$-iC_3 \hat{\zeta}_{\tilde{x}}^0 = \frac{1}{\lambda^2} \hat{\zeta}_{\tilde{x}}^0, \quad (177)$$

and thus $C_3 = -1/(i\lambda^2)$.

The general solution of $\hat{\tilde{u}}^0$ is thus

$$\hat{\tilde{u}}^0 = C_1 e^{\tilde{r}_{M_2} \tilde{z}} + C_2 e^{-\tilde{r}_{M_2} \tilde{z}} - \frac{1}{i\lambda^2} \hat{\zeta}_{\tilde{x}}^0. \quad (178)$$

To obtain the coefficients C_1 and C_2 , the boundary conditions (170) and (171) are used. Substituting Eq. (178) into Eq. (170) yields

$$A_v(\tilde{r}_{M_2} C_1 - \tilde{r}_{M_2} C_2) = 0, \quad (179)$$

implying that $C_1 = C_2 = C$. Subsequently, substituting Eq. (178) into Eq. (171) and solving for C leads to

$$C = \frac{1}{2} \frac{1}{i\lambda^2} \frac{\theta}{\tilde{r}_{M_2} \sinh(\tilde{r}_{M_2} \tilde{H}) + \theta \cosh(\tilde{r}_{M_2} \tilde{H})} \hat{\zeta}_{\tilde{x}}^0. \quad (180)$$

Substituting Eq. (180) into the general equation for $\hat{\tilde{u}}^0$, Eq. (178), and rewriting in terms of hyperbolic functions results in the expression for the horizontal velocity amplitude $\hat{\tilde{u}}^0$, i.e.

$$\hat{\tilde{u}}^0 = \frac{1}{i\lambda^2} \hat{\zeta}_{\tilde{x}}^0 \left(\tilde{\alpha}_{M_2} \cosh(\tilde{r}_{M_2} \tilde{z}) - 1 \right), \quad (181)$$

with

$$\tilde{\alpha}_{M_2}(\tilde{x}) = \frac{\theta}{\tilde{r}_{M_2} \sinh(\tilde{r}_{M_2} \tilde{H}) + \theta \cosh(\tilde{r}_{M_2} \tilde{H})}. \quad (182)$$

Surface elevation

Next, the solution (166) along with the solution for $\hat{\tilde{u}}^0$ are substituted into the depth-averaged continuity equation (169). With the third term on the left-hand side of Eq. (169)

$$\frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \int_{-\tilde{H}}^0 \hat{u}^0 d\tilde{z} = \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \left[-\frac{1}{i\lambda^2} \hat{\zeta}_{\tilde{x}}^0 \left(\tilde{z} - \frac{\tilde{\alpha}_{M_2}}{\tilde{r}_{M_2}} \sinh(\tilde{r}_{M_2} \tilde{z}) \right) \right]_{-\tilde{H}}^0, \quad (183)$$

$$= \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \frac{1}{i\lambda^2} \hat{\zeta}_{\tilde{x}}^0 \left(\frac{\tilde{\alpha}_{M_2}}{\tilde{r}_{M_2}} \sinh(\tilde{r}_{M_2} \tilde{H}) - \tilde{H} \right), \quad (184)$$

and the second term on the left-hand side of Eq. (169)

$$\left(\int_{-\tilde{H}}^0 \hat{u}^0 d\tilde{z} \right)_{\tilde{x}} = \left(\frac{1}{i\lambda^2} \hat{\zeta}_{\tilde{x}}^0 \left(\frac{\tilde{\alpha}_{M_2}}{\tilde{r}_{M_2}} \sinh(\tilde{r}_{M_2} \tilde{H}) - \tilde{H} \right) \right)_{\tilde{x}}, \quad (185)$$

$$= -\frac{1}{i\lambda^2} \left[\hat{\zeta}_{\tilde{x}\tilde{x}}^0 \tilde{H} + \hat{\zeta}_{\tilde{x}}^0 \tilde{H}_{\tilde{x}} - \hat{\zeta}_{\tilde{x}\tilde{x}}^0 \frac{\tilde{\alpha}_{M_2}}{\tilde{r}_{M_2}} \sinh(\tilde{r}_{M_2} \tilde{H}) \right] \quad (186)$$

$$- \hat{\zeta}_{\tilde{x}}^0 \left(\frac{\tilde{\alpha}_{M_2, \tilde{x}} \tilde{r}_{M_2} - \tilde{\alpha}_{M_2} \tilde{r}_{M_2, \tilde{x}}}{\tilde{r}_{M_2}^2} \sinh(\tilde{r}_{M_2} \tilde{H}) + \frac{\tilde{\alpha}_{M_2}}{\tilde{r}_{M_2}} \cosh(\tilde{r}_{M_2} \tilde{H}) (\tilde{r}_{M_2} \tilde{H}_{\tilde{x}} + \tilde{r}_{M_2, \tilde{x}} \tilde{H}) \right) \Bigg]. \quad (187)$$

Eq. (169) then gives a second-order linear ordinary differential equation (ODE) as a function of $\hat{\zeta}^0$

$$\tilde{T}_1 \hat{\zeta}_{\tilde{x}\tilde{x}}^0 + \tilde{T}_2 \hat{\zeta}_{\tilde{x}}^0 - \lambda^2 \hat{\zeta}^0 = 0, \quad (188)$$

with

$$\tilde{T}_1 = \frac{\tilde{\alpha}_{M_2}}{\tilde{r}_{M_2}} \sinh(\tilde{r}_{M_2} \tilde{H}) - \tilde{H}, \quad (189)$$

$$\begin{aligned} \tilde{T}_2 = & \left(\frac{\tilde{\alpha}_{M_2} \tilde{r}_{M_2, \tilde{x}} \tilde{H}}{\tilde{r}} + \tilde{\alpha}_{M_2} \tilde{H}_{\tilde{x}} \right) \cosh(\tilde{r}_{M_2} \tilde{H}) - \frac{1}{\tilde{r}_{M_2}} \left(-\frac{\tilde{\alpha}_{M_2} \tilde{B}_{\tilde{x}}}{\tilde{B}} - \tilde{\alpha}_{M_2, \tilde{x}} + \frac{\tilde{\alpha}_{M_2} \tilde{r}_{M_2, \tilde{x}}}{\tilde{r}_{M_2}} \right) \sinh(\tilde{r}_{M_2} \tilde{H}) \\ & - \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \tilde{H} - \tilde{H}_{\tilde{x}}. \end{aligned} \quad (190)$$

Eq. (188) is subject to the boundary conditions at the seaward, $\tilde{x} = 0$, and landward side of the estuary, $\tilde{x} = 1$,

$$\hat{\zeta}^0(0) = 1, \quad (191)$$

$$\int_{-\tilde{H}}^0 \hat{u}^0(1) d\tilde{z} = 0. \quad (192)$$

Eq. (188) generally needs to be solved numerically because of the non-constant coefficients in the ODE. A special case in which an analytical solution for $\hat{\zeta}^0$ can be found is when $\tilde{\alpha}_{M_2}$, $\tilde{r}_{M_2}$, $\tilde{H}$ and $\tilde{B}_{\tilde{x}}/\tilde{B}$ are uniform in the $\tilde{x}$ -direction and thus their derivatives w.r.t. $\tilde{x}$ are zero. After rearranging terms, Eq. (188) then reduces to

$$\hat{\zeta}_{\tilde{x}\tilde{x}}^0 + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}}\hat{\zeta}_{\tilde{x}}^0 - \frac{\lambda^2}{\frac{\tilde{\alpha}_{M_2}}{\tilde{r}_{M_2}} \sinh(\tilde{r}_{M_2}\tilde{H}) - \tilde{H}}\hat{\zeta}^0 = 0. \quad (193)$$

Assuming $\hat{\zeta}^0 = e^{p\tilde{x}}$ to be a solution to Eq. (193), it follows that p must be a root of the characteristic equation

$$p^2 + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}}p - \frac{\lambda^2}{\frac{\tilde{\alpha}_{M_2}}{\tilde{r}_{M_2}} \sinh(\tilde{r}_{M_2}\tilde{H}) - \tilde{H}} = 0, \quad (194)$$

which has solutions

$$p_{1,2} = \frac{-\frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \pm \sqrt{\left(\frac{\tilde{B}_{\tilde{x}}}{\tilde{B}}\right)^2 + \frac{4\lambda^2}{\frac{\tilde{\alpha}_{M_2}}{\tilde{r}_{M_2}} \sinh(\tilde{r}_{M_2}\tilde{H}) - \tilde{H}}}}{2}. \quad (195)$$

The general solution of $\hat{\zeta}^0$ thus reads

$$\hat{\zeta}^0 = c_1 e^{p_1 \tilde{x}} + c_2 e^{p_2 \tilde{x}}. \quad (196)$$

Using the boundary condition at the seaward side of the estuary, Eq. (191), it follows that $c_1 + c_2 = 1$. From the landward boundary condition, Eq. (192), it follows that $\hat{\zeta}_{\tilde{x}}^0(1) = 0$. Setting the derivative of Eq. (196) equal to zero, leads to the following expressions for c_1 and c_2 :

$$c_1 = -\frac{p_2 e^{p_2}}{p_1 e^{p_1} - p_2 e^{p_2}}, \quad (197)$$

$$c_2 = \frac{p_1 e^{p_1}}{p_1 e^{p_1} - p_2 e^{p_2}}. \quad (198)$$

The solution for $\hat{\zeta}^0$ is

$$\hat{\zeta}^0 = -\frac{p_2 e^{p_2}}{p_1 e^{p_1} - p_2 e^{p_2}} e^{p_1 \tilde{x}} + \frac{p_1 e^{p_1}}{p_1 e^{p_1} - p_2 e^{p_2}} e^{p_2 \tilde{x}}. \quad (199)$$

For readability Eq. (199) can be rewritten in hyperbolic form. Introducing

$$p_{1,2} = -\beta \pm \beta\Gamma, \quad (200)$$

with

$$\beta = \frac{1}{2} \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}}, \quad (201)$$

$$\Gamma = \sqrt{1 + \frac{\lambda^2}{\beta^2 \left(\frac{\tilde{\alpha}_{M_2}}{\tilde{r}_{M_2}} \sinh(\tilde{r}_{M_2} \tilde{H}) - \tilde{H} \right)}}, \quad (202)$$

the hyperbolic expression for the surface elevation $\hat{\zeta}^0$ of a converging channel with constant coefficients is

$$\hat{\zeta}^0 = \left[\frac{\Gamma \cosh(\beta \Gamma(\tilde{x} - 1)) + \sinh(\beta \Gamma(\tilde{x} - 1))}{\Gamma \cosh(\beta \Gamma) - \sinh(\beta \Gamma)} \right] e^{-\beta \tilde{x}}. \quad (203)$$

Vertical velocity

Finally, the vertical velocity $\hat{w}^0$ can be found by substituting the expression for $\hat{u}^0$ into the continuity equation (168). This results in an expression for $\hat{w}_{\tilde{z}}^0$:

$$\hat{w}_{\tilde{z}}^0 = \frac{1}{i\lambda^2} \left(1 - \tilde{\alpha}_{M_2} \cosh(\tilde{r}_{M_2} \tilde{z}) \right) \hat{\zeta}_{\tilde{x}\tilde{x}}^0 - \frac{1}{i\lambda^2} \left(\left(\tilde{\alpha}_{M_2, \tilde{x}} + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \tilde{\alpha}_{M_2} \right) \cosh(\tilde{r}_{M_2} \tilde{z}) + \tilde{\alpha}_{M_2} \tilde{r}_{M_2, \tilde{x}} \tilde{z} \sinh(\tilde{r}_{M_2} \tilde{z}) + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \right) \hat{\zeta}_{\tilde{x}}^0. \quad (204)$$

Integrating Eq. (204) results in

$$\begin{aligned} \hat{w}^0 &= \frac{1}{i\lambda^2} \left(\tilde{z} - \frac{\tilde{\alpha}_{M_2}}{\tilde{r}_{M_2}} \sinh(\tilde{r}_{M_2} \tilde{z}) \right) \hat{\zeta}_{\tilde{x}\tilde{x}}^0 \\ &\quad - \frac{1}{i\lambda^2 \tilde{r}_{M_2}} \left(\left(\tilde{\alpha}_{M_2, \tilde{x}} - \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \tilde{\alpha}_{M_2} - \frac{\tilde{\alpha}_{M_2} \tilde{r}_{M_2, \tilde{x}}}{\tilde{r}_{M_2}} \right) \sinh(\tilde{r}_{M_2} \tilde{z}) + \tilde{\alpha}_{M_2} \tilde{r}_{M_2, \tilde{x}} \tilde{z} \cosh(\tilde{r}_{M_2} \tilde{z}) + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \tilde{r}_{M_2} \tilde{z} \right) \hat{\zeta}_{\tilde{x}}^0 + I, \end{aligned} \quad (205)$$

where I is an integration constant that can be determined using the kinematic boundary condition at the free surface $\tilde{z} = 0$

$$\hat{w}^0(\tilde{x}, 0) = i\hat{\zeta}^0. \quad (206)$$

It follows that $I = i\hat{\zeta}^0$ and, consequently, the expression for the vertical velocity amplitude is

$$\hat{w}^0 = \tilde{T}'_1 \hat{\zeta}_{\tilde{x}\tilde{x}}^0 + \tilde{T}'_2 \hat{\zeta}_{\tilde{x}}^0 - \lambda^2 \hat{\zeta}^0, \quad (207)$$

with

$$\tilde{T}'_1 = \tilde{z} - \frac{\tilde{\alpha}_{M_2}}{\tilde{r}_{M_2}} \sinh(\tilde{r}_{M_2} \tilde{z}), \quad (208)$$

$$\tilde{T}'_2 = -\frac{1}{\tilde{r}_{M_2}} \left(\left(\tilde{\alpha}_{M_2, \tilde{x}} - \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \tilde{\alpha}_{M_2} - \frac{\tilde{\alpha}_{M_2} \tilde{r}_{M_2, \tilde{x}}}{\tilde{r}_{M_2}} \right) \sinh(\tilde{r}_{M_2} \tilde{z}) + \tilde{\alpha}_{M_2} \tilde{r}_{M_2, \tilde{x}} \tilde{z} \cosh(\tilde{r}_{M_2} \tilde{z}) + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \tilde{r}_{M_2} \tilde{z} \right). \quad (209)$$

A1.2.2 First order solutions

For ease of reference, the first order system of equations is repeated here

$$\delta \tilde{u}_{\tilde{z}\tilde{z}}^1 - \tilde{u}_{\tilde{t}}^1 = \frac{1}{\lambda^2} \tilde{\zeta}_{\tilde{x}}^1 + \tilde{\xi} - \mu \langle \tilde{s}_{\tilde{x}} \rangle \tilde{z}, \quad (210)$$

$$\tilde{u}_{\tilde{x}}^1 + \tilde{w}_{\tilde{z}}^1 + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \tilde{u}^1 = 0, \quad (211)$$

$$\tilde{\zeta}_{\tilde{t}}^1 + \left(\frac{\partial}{\partial \tilde{x}} + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \right) \left(\int_{-\tilde{H}}^0 \tilde{u}^1 d\tilde{z} + \tilde{\gamma} \right) = 0, \quad (212)$$

where we have introduced the following simplifying dimensionless quantities

$$\tilde{\xi}(\tilde{x}, \tilde{z}, \tilde{t}) = \tilde{u}^0(\tilde{x}, \tilde{z}, \tilde{t}) \tilde{u}_{\tilde{x}}^0(\tilde{x}, \tilde{z}, \tilde{t}) + \tilde{w}^0(\tilde{x}, \tilde{z}, \tilde{t}) \tilde{u}_{\tilde{z}}^0(\tilde{x}, \tilde{z}, \tilde{t}), \quad (213)$$

$$\tilde{\gamma}(\tilde{x}, \tilde{t}) = \tilde{\zeta}^0(\tilde{x}, \tilde{t}) \tilde{u}^0(\tilde{x}, \tilde{t}) \Big|_{\tilde{z}=0}, \quad (214)$$

$$\tilde{\chi}(\tilde{x}, \tilde{t}) = \tilde{\zeta}^0(\tilde{x}, \tilde{t}) \tilde{u}_{\tilde{z}\tilde{z}}^0(\tilde{x}, \tilde{t}) \Big|_{\tilde{z}=0}. \quad (215)$$

Here, $\tilde{\xi}$ represents advection of momentum and $\tilde{\gamma}$ is related to the tidal return or stokes flow, which is the result of a positive correlation between the zeroth order vertical and horizontal tide. Finally, $\tilde{\chi}$ originates from the first order contribution of the stress free boundary condition.

The corresponding boundary conditions are

$$\tilde{\zeta}^1 = \frac{A_{M_4}}{A_{M_2}} \cos(2\tilde{t} - \varphi), \quad \text{at } \tilde{x} = 0, \quad (216)$$

$$\int_{-\tilde{H}}^0 \tilde{u}^1 d\tilde{z} + \tilde{\gamma} = -\frac{Q}{B_0 H_0 U \tilde{B}}, \quad \text{at } \tilde{x} = 1, \quad (217)$$

$$\tilde{w}^1 = \tilde{\zeta}_{\tilde{t}}^1 + \tilde{\zeta}_{\tilde{x}}^0 \tilde{u}^0 - \tilde{\zeta}^0 \tilde{w}_{\tilde{z}}^0, \quad \text{at } \tilde{z} = 0, \quad (218)$$

$$A_v \tilde{u}_{\tilde{z}}^1 = -A_v \tilde{\chi}, \quad \text{at } \tilde{z} = 0, \quad (219)$$

$$\tilde{u}_{\tilde{z}}^1 = \theta \tilde{u}^1, \quad \text{at } \tilde{z} = -\tilde{H}, \quad (220)$$

$$\tilde{w}^1 = -\tilde{u}^1 \tilde{H}_{\tilde{x}}, \quad \text{at } \tilde{z} = -\tilde{H}. \quad (221)$$

As is apparent from Eq. (210)-(221), the first order velocity and water level are forced externally by an M_4 tidal component and a constant river discharge and internally by the salinity gradient, the leading order advection, and tidal return flow (stokes flow). Since we assumed solutions with a M_2 tidal frequency, Eq. (166), the product of two leading order forcing terms, Eqs. (213)-(215), with an M_2 frequency result in a residual (M_0) and a M_4 frequency as follows. The solutions, Eq. (166), can be written in the following form

$$(\tilde{u}^0, \tilde{w}^0, \tilde{\zeta}^0) = \frac{1}{2} \left(\hat{u}^0(\tilde{x}, \tilde{z}), \hat{w}^0(\tilde{x}, \tilde{z}), \hat{\zeta}^0(\tilde{x}) \right) e^{i\tilde{t}} + \frac{1}{2} \left(\hat{u}^{0*}(\tilde{x}, \tilde{z}), \hat{w}^{0*}(\tilde{x}, \tilde{z}), \hat{\zeta}^{0*}(\tilde{x}) \right) e^{-i\tilde{t}}, \quad (222)$$

where the superscript $(.)^*$ denotes the complex conjugate (c.c.) of that quantity. The product of two leading order variables appearing in the forcing terms Eqs. (213)-(215), e.g. $\tilde{\gamma} = \tilde{\zeta}^0 \tilde{u}^0|_{\tilde{z}=0}$, thus result in a residual (M_0) and a M_4 contribution

$$\begin{aligned} \tilde{\gamma} &= \frac{1}{4} \left[\hat{\zeta}^0 e^{i\tilde{t}} + \hat{\zeta}^{0*} e^{-i\tilde{t}} \right] \left[\hat{u}^0|_{\tilde{z}=0} e^{i\tilde{t}} + \hat{u}^{0*}|_{\tilde{z}=0} e^{-i\tilde{t}} \right], \\ &= \frac{1}{4} \underbrace{\left[\hat{\zeta}^0 \hat{u}^{0*}|_{\tilde{z}=0} + \hat{\zeta}^{0*} \hat{u}^0|_{\tilde{z}=0} \right]}_{M_0} + \frac{1}{4} \underbrace{\left[\hat{\zeta}^0 \hat{u}^0|_{\tilde{z}=0} e^{2i\tilde{t}} + \hat{\zeta}^{0*} \hat{u}^{0*}|_{\tilde{z}=0} e^{-2i\tilde{t}} \right]}_{M_4}, \end{aligned} \quad (223)$$

$$= \langle \tilde{\gamma} \rangle + [\tilde{\gamma}], \quad (224)$$

In the following analysis we will derive the equations and solutions for the residual and M_4 contributions to the surface elevation and the vertical and horizontal velocities. Thereby, we will use $\langle . \rangle$ and $[.]$ to denote the residual (M_0) or time-averaged and M_4 contribution, respectively.

Residual flow velocity (M_0)

The equations for the residual flow are obtained by taking the tide-averaged component of the first order equations. The momentum equation with its boundary conditions is then given by

$$\hat{u}_{\tilde{z}\tilde{z}}^{10} = \frac{1}{\delta \lambda^2} \hat{\zeta}_{\tilde{x}}^{10} + \frac{1}{\delta} \langle \hat{\xi} \rangle - \frac{\mu}{\delta} \langle \hat{s}_{\tilde{x}} \rangle \tilde{z}, \quad (225)$$

$$\hat{\zeta}^{10} = 0, \quad \text{at } \tilde{x} = 0, \quad (226)$$

$$\int_{-\tilde{H}}^0 \hat{u}^{10} d\tilde{z} = -\frac{Q}{B_0 H_0 U \tilde{B}} - \langle \hat{\gamma} \rangle, \quad \text{at } \tilde{x} = 1, \quad (227)$$

$$\hat{u}_{\tilde{z}}^{10} = -\langle \hat{\chi} \rangle, \quad \text{at } \tilde{z} = 0, \quad (228)$$

$$\hat{u}_{\tilde{z}}^{10} = \theta \hat{u}^{10}, \quad \text{at } \tilde{z} = -\tilde{H}, \quad (229)$$

where the first number in the superscript refers to the order and the second number to the frequency. Since the solution to $\hat{u}^{10}$ is linear, it can be constructed by adding the contributions of the different forcing terms to the residual velocity, i.e.

$$\hat{u}^{10} = \tilde{u}^{\text{salt}} + \tilde{u}^{\text{adv}} + \tilde{u}^{\text{no-stress}} + \tilde{u}^{\text{river}} + \tilde{u}^{\text{stokes}}. \quad (230)$$

The expressions for the contribution of each forcing term to the residual velocity can be derived by taking into account only the terms in the momentum equation and appropriate boundary conditions for each contribution. For the remainder of this appendix, the contribution name in the superscript of a quantity denotes its (dimensionless) complex amplitude.

Salinity field For the salinity field contribution, the equations become

$$\tilde{u}_{\tilde{z}\tilde{z}}^{\text{salt}} = \frac{1}{\delta\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{salt}} - \frac{\mu}{\delta} \langle \hat{s}_{\tilde{x}} \rangle \tilde{z}, \quad (231)$$

$$\tilde{\zeta}^{\text{salt}} = 0, \quad \text{at } \tilde{x} = 0, \quad (232)$$

$$\int_{-\tilde{H}}^0 \tilde{u}^{\text{salt}} d\tilde{z} = 0, \quad \text{at } \tilde{x} = 1, \quad (233)$$

$$\tilde{u}_{\tilde{z}}^{\text{salt}} = 0, \quad \text{at } \tilde{z} = 0, \quad (234)$$

$$\tilde{u}_{\tilde{z}}^{\text{salt}} = \theta \tilde{u}^{\text{salt}}, \quad \text{at } \tilde{z} = -\tilde{H}. \quad (235)$$

Integrating Eq. (231) twice leads to

$$\tilde{u}_{\tilde{z}}^{\text{salt}} = \frac{1}{\delta\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{salt}} \tilde{z} - \frac{1}{2} \frac{\mu}{\delta} \langle \hat{s}_{\tilde{x}} \rangle \tilde{z}^2 + C^{\text{salt}}, \quad (236)$$

$$\tilde{u}^{\text{salt}} = \frac{1}{2} \frac{1}{\delta\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{salt}} \tilde{z}^2 - \frac{1}{6} \frac{\mu}{\delta} \langle \hat{s}_{\tilde{x}} \rangle \tilde{z}^3 + C^{\text{salt}} \tilde{z} + D^{\text{salt}}, \quad (237)$$

where $C^{\text{salt}}(\tilde{x})$ and $D^{\text{salt}}(\tilde{x})$ are integration constants to be determined using the boundary conditions. From the boundary condition at $\tilde{z} = 0$, Eq. (234), it follows that $C^{\text{salt}} = 0$. Substituting Eqs. (236) and (237) into the boundary condition at $\tilde{z} = -\tilde{H}$, Eq. (235), and using $C^{\text{salt}} = 0$, results in the following expression for D^{salt} :

$$D^{\text{salt}} = -\frac{\tilde{H}}{\delta\lambda^2} \left(\frac{1}{\theta} + \frac{\tilde{H}}{2} \right) \tilde{\zeta}_{\tilde{x}}^{\text{salt}} - \frac{\mu\tilde{H}^2}{2\delta} \left(\frac{1}{\theta} + \frac{\tilde{H}}{3} \right) \langle \hat{s}_{\tilde{x}} \rangle. \quad (238)$$

Substituting $C^{\text{salt}} = 0$ and D^{salt} , Eq. (238), into Eq. (237) results in

$$\tilde{u}^{\text{salt}} = \frac{1}{\delta} \left[\frac{1}{\lambda^2} \left(\frac{\tilde{z}^2}{2} - \frac{\tilde{H}^2}{2} - \frac{\tilde{H}}{\theta} \right) \tilde{\zeta}_{\tilde{x}}^{\text{salt}} - \left(\frac{\tilde{z}^3}{6} + \frac{\tilde{H}^3}{6} + \frac{\tilde{H}^2}{2\theta} \right) \mu \langle \hat{s}_{\tilde{x}} \rangle \right]. \quad (239)$$

The expression for $\tilde{\zeta}_{\tilde{x}}^{\text{salt}}$ is obtained by substituting Eq. (239) into the boundary condition at $\tilde{x} = 1$, Eq. (233). Integrating Eq. (239) and equalling to zero leads to

$$\tilde{\zeta}_{\tilde{x}}^{\text{salt}} = -\frac{\lambda^2 \mu \langle \hat{s}_{\tilde{x}} \rangle \left(\frac{\tilde{H}^4}{8} + \frac{\tilde{H}^3}{2\theta} \right)}{\left(\frac{\tilde{H}^3}{3} + \frac{\tilde{H}^2}{\theta} \right)}. \quad (240)$$

This is also referred to as gravitational circulation. Numerically integrating Eq. (240) with respect to $\tilde{x}$ leads to an expression for the water level set-up $\tilde{\zeta}^{\text{salt}}$ due to the the baroclinic pressure gradient and the constraint of no net transport.

Advection of momentum: For the advection of momentum contribution, the equations become

$$\tilde{u}_{\tilde{z}\tilde{z}}^{\text{adv}} = \frac{1}{\delta\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{adv}} + \frac{1}{\delta} \langle \hat{\xi} \rangle, \quad (241)$$

$$\tilde{\zeta}^{\text{adv}} = 0, \quad \text{at } \tilde{x} = 0, \quad (242)$$

$$\int_0^{\tilde{H}} \tilde{u}^{\text{adv}} d\tilde{z} = 0, \quad \text{at } \tilde{x} = 1, \quad (243)$$

$$\tilde{u}_{\tilde{z}}^{\text{adv}} = 0, \quad \text{at } \tilde{z} = 0, \quad (244)$$

$$\tilde{u}_{\tilde{z}}^{\text{adv}} = \theta \tilde{u}^{\text{adv}}, \quad \text{at } \tilde{z} = -\tilde{H}. \quad (245)$$

Since the advection term $\hat{\xi}$ on the right-hand side of Eq. (241) is a function of $\tilde{z}$, we use the method of variation of parameters to find a solution for $\tilde{u}^{\text{adv}}$. First, we must solve the homogeneous equation $\tilde{u}_{\tilde{z}\tilde{z}}^{\text{adv}} = 0$, which leads to

$$\tilde{u}_h^{\text{adv}} = C^{\text{adv,h}} \cdot \tilde{z} + D^{\text{adv,h}}, \quad (246)$$

where $C^{\text{adv,h}}(\tilde{x})$ and $D^{\text{adv,h}}(\tilde{x})$ are integration constants. Second, we solve the particular equation for the water level forcing, $\tilde{u}_{\tilde{z}\tilde{z}}^{\text{adv}} = \tilde{\zeta}_{\tilde{x}}^{\text{adv}}/\delta\lambda^2$, which leads to

$$\tilde{u}_{p,\tilde{z}}^{\text{adv}} = \frac{1}{2} \frac{1}{\delta\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{adv}} \tilde{z}^2 + C^{\text{adv,p}} \tilde{z} + D^{\text{adv,p}}, \quad (247)$$

where $C^{\text{adv,p}}(\tilde{x})$ and $D^{\text{adv,p}}(\tilde{x})$ are integration constants. Third, we solve the particular equation for the advective forcing, $\tilde{u}_{\tilde{z}\tilde{z}}^{\text{adv}} = \langle \hat{\xi} \rangle/\delta$. Since $\hat{\xi}$ is a function of $\tilde{z}$, we seek a pair of functions, $f(\tilde{x}, \tilde{z})$ and $g(\tilde{x}, \tilde{z})$, so that the particular solution reads

$$\tilde{u}_{p,\tilde{\xi}}^{\text{adv}} = f\tilde{z} + g, \quad (248)$$

which has the same form as the homogeneous solution, Eq. (246). The first derivative of $\tilde{u}_{p,\tilde{\xi}}^{\text{adv}}$ with respect to $\tilde{z}$ is

$$(\tilde{u}_{p,\tilde{\xi}}^{\text{adv}})_{\tilde{z}} = f_{\tilde{z}}\tilde{z} + f + g_{\tilde{z}}. \quad (249)$$

Now, let us assume that whatever f and g are, they will satisfy the following

$$f_{\tilde{z}}\tilde{z} + g_{\tilde{z}} = 0. \quad (250)$$

The first derivative $(\tilde{u}_{p,\tilde{\xi}}^{\text{adv}})_{\tilde{z}}$ is now equal to f and the second derivative $(\tilde{u}_{p,\tilde{\xi}}^{\text{adv}})_{\tilde{z}\tilde{z}}$ equals $f_{\tilde{z}}$. Plugging the latter into the original momentum equation, Eq. (241), gives

$$f_{\tilde{z}} = \frac{1}{\delta} \langle \hat{\xi} \rangle. \quad (251)$$

Substituting Eq. (251) into Eq. (250) gives

$$g_{\tilde{z}} = -\frac{1}{\delta} \langle \hat{\xi} \rangle \tilde{z}. \quad (252)$$

Integrating Eqs. (251) and (252) and substituting into Eq. (248) leads to

$$\tilde{u}_{p,\tilde{\xi}}^{\text{adv}} = \frac{1}{\delta} \left[\tilde{z} \int_{-\tilde{H}}^{\tilde{z}} \langle \hat{\xi} \rangle d\tilde{z}' - \int_{-\tilde{H}}^{\tilde{z}} \langle \hat{\xi} \rangle \tilde{z}' d\tilde{z}' \right]. \quad (253)$$

Adding Eqs. (246), (247) and (253), results in the full expression for $\tilde{u}^{\text{adv}}$, viz.

$$\tilde{u}^{\text{adv}} = \frac{1}{\delta} \left[\frac{1}{2\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{adv}} \tilde{z}^2 + C^{\text{adv}} \cdot \tilde{z} + D^{\text{adv}} + \tilde{z} \int_{-\tilde{H}}^{\tilde{z}} \langle \hat{\xi} \rangle d\tilde{z}' - \int_{-\tilde{H}}^{\tilde{z}} \langle \hat{\xi} \rangle \tilde{z}' d\tilde{z}' \right], \quad (254)$$

where $C^{\text{adv}} = (C^{\text{adv,h}} + C^{\text{adv,p}})\delta$ and $D^{\text{adv}} = (D^{\text{adv,h}} + D^{\text{adv,p}})\delta$. From the boundary condition at $\tilde{z} = 0$, Eq. (244), it follows that

$$C^{\text{adv}}(\tilde{x}) = - \int_{-\tilde{H}}^0 \langle \hat{\xi} \rangle d\tilde{z}. \quad (255)$$

Substituting Eq. (254) and Eq. (255) into the boundary condition at $\tilde{z} = -\tilde{H}$, Eq. (245), results in the following expression for D^{adv}

$$D^{\text{adv}}(\tilde{x}) = -\frac{\tilde{H}}{\lambda^2} \left[\frac{1}{\theta} + \frac{\tilde{H}}{2} \right] \tilde{\zeta}_{\tilde{x}}^{\text{adv}} - \left[\frac{1}{\theta} + \tilde{H} \right] \int_{-\tilde{H}}^0 \langle \hat{\xi} \rangle d\tilde{z}. \quad (256)$$

Substituting the expressions for C^{adv} and D^{adv} into Eq. (254) results in

$$\tilde{u}^{\text{adv}} = \frac{1}{\delta} \left[\frac{1}{\lambda^2} \left(\frac{\tilde{z}^2}{2} - \frac{\tilde{H}^2}{2} - \frac{\tilde{H}}{\theta} \right) \tilde{\zeta}_{\tilde{x}}^{\text{adv}} - \left(\tilde{z} + \tilde{H} + \frac{1}{\theta} \right) \int_{-\tilde{H}}^0 \langle \hat{\xi} \rangle d\tilde{z} + \tilde{z} \int_{-\tilde{H}}^{\tilde{z}} \langle \hat{\xi} \rangle d\tilde{z}' - \int_{-\tilde{H}}^{\tilde{z}} \langle \hat{\xi} \rangle \tilde{z}' d\tilde{z}' \right]. \quad (257)$$

The expression for $\tilde{\zeta}_{\tilde{x}}^{\text{adv}}$ is obtained by substituting Eq. (257) into the boundary condition at $\tilde{x} = 1$, Eq. (243). Integrating Eq. (257) and equalling to zero leads to

$$\tilde{\zeta}_{\tilde{x}}^{\text{adv}} = \frac{\lambda^2 \int_{-\tilde{H}}^0 \left[\tilde{z} \int_{-\tilde{H}}^{\tilde{z}} \langle \hat{\xi} \rangle d\tilde{z}' - \int_{-\tilde{H}}^{\tilde{z}} \langle \hat{\xi} \rangle \tilde{z}' d\tilde{z}' \right] d\tilde{z} - \lambda^2 \left(\frac{\tilde{H}^2}{2} + \frac{\tilde{H}}{\theta} \right) \int_{-\tilde{H}}^0 \langle \hat{\xi} \rangle d\tilde{z}}{\left(\frac{\tilde{H}^3}{3} + \frac{\tilde{H}^2}{\theta} \right)}. \quad (258)$$

Numerically integrating Eq. (258) with respect to $\tilde{x}$ leads to an expression for the water level set-up or set-down $\tilde{\zeta}^{\text{adv}}$ due to the advection of momentum.

Stress free boundary condition: For the contribution of the stress free boundary condition, the equations become

$$\tilde{u}_{\tilde{z}\tilde{z}}^{\text{no-stress}} = \frac{1}{\delta\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{no-stress}}, \quad (259)$$

$$\tilde{\zeta}^{\text{no-stress}} = 0, \quad \text{at } \tilde{x} = 0, \quad (260)$$

$$\int_{-\tilde{H}}^0 \tilde{u}^{\text{no-stress}} d\tilde{z} = 0, \quad \text{at } \tilde{x} = 1, \quad (261)$$

$$\tilde{u}_{\tilde{z}}^{\text{no-stress}} = -\langle \hat{\chi} \rangle, \quad \text{at } \tilde{z} = 0, \quad (262)$$

$$\tilde{u}_{\tilde{z}}^{\text{no-stress}} = \theta \tilde{u}^{\text{no-stress}}, \quad \text{at } \tilde{z} = -\tilde{H}. \quad (263)$$

Integrating Eq. (259) twice leads to

$$\tilde{u}_{\tilde{z}}^{\text{no-stress}} = \frac{1}{\delta\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{no-stress}} \tilde{z} + C^{\text{no-stress}}, \quad (264)$$

$$\tilde{u}^{\text{no-stress}} = \frac{1}{2} \frac{1}{\delta\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{no-stress}} \tilde{z}^2 + C^{\text{no-stress}} \tilde{z} + D^{\text{no-stress}}, \quad (265)$$

where $C^{\text{no-stress}}(\tilde{x})$ and $D^{\text{no-stress}}(\tilde{x})$ are integration constants to be determined using the boundary conditions. From the boundary condition at $\tilde{z} = 0$, Eq. (262), it follows that

$$C^{\text{no-stress}} = -\langle \hat{\chi} \rangle. \quad (266)$$

Substituting Eqs. (264) and (265) into the boundary condition at $\tilde{z} = -\tilde{H}$, Eq. (263), and using the expression for $C^{\text{no-stress}}$, yields

$$D^{\text{no-stress}} = -\frac{\tilde{H}}{\delta\lambda^2} \left(\frac{1}{\theta} + \frac{\tilde{H}}{2} \right) \tilde{\zeta}_{\tilde{x}}^{\text{no-stress}} - \left(\frac{1}{\theta} + \tilde{H} \right) \langle \hat{\chi} \rangle. \quad (267)$$

Substituting the expressions for $C^{\text{no-stress}}$ and $D^{\text{no-stress}}$ into Eq. (265) results in

$$\tilde{u}^{\text{no-stress}} = \frac{1}{\delta\lambda^2} \left(\frac{\tilde{z}^2}{2} - \frac{\tilde{H}^2}{2} - \frac{\tilde{H}}{\theta} \right) \tilde{\zeta}_{\tilde{x}}^{\text{no-stress}} - \left(\tilde{z} + \tilde{H} + \frac{1}{\theta} \right) \langle \hat{\chi} \rangle. \quad (268)$$

The expression for $\tilde{\zeta}_{\tilde{x}}^{\text{no-stress}}$ is obtained by substituting Eq. (268) into the boundary condition at $\tilde{x} = 1$, Eq. (261). Integrating Eq. (268) and equalling to zero leads to

$$\tilde{\zeta}_{\tilde{x}}^{\text{no-stress}} = -\frac{\delta\lambda^2 \langle \hat{\chi} \rangle \left(\frac{\tilde{H}^2}{2} + \frac{\tilde{H}}{\theta} \right)}{\left(\frac{\tilde{H}^3}{3} + \frac{\tilde{H}^2}{\theta} \right)}. \quad (269)$$

Numerically integrating Eq. (269) with respect to $\tilde{x}$ leads to an expression for the water level set-up or set-down $\tilde{\zeta}^{\text{no-stress}}$ due to the stress free boundary condition.

River outflow: For the contribution of the river flow, the equations become

$$\tilde{u}_{\tilde{z}\tilde{z}}^{\text{river}} = \frac{1}{\delta\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{river}}, \quad (270)$$

$$\tilde{\zeta}^{\text{river}} = 0, \quad \text{at } \tilde{x} = 0, \quad (271)$$

$$\int_{-\tilde{H}}^0 \tilde{u}^{\text{river}} d\tilde{z} = -\frac{Q}{B_0 H_0 U \tilde{B}}, \quad \text{at } \tilde{x} = 1, \quad (272)$$

$$\tilde{u}_{\tilde{z}}^{\text{river}} = 0, \quad \text{at } \tilde{z} = 0, \quad (273)$$

$$\tilde{u}_{\tilde{z}}^{\text{river}} = \theta \tilde{u}^{\text{river}}, \quad \text{at } \tilde{z} = -\tilde{H}. \quad (274)$$

Integrating Eq. (270) twice leads to

$$\tilde{u}_{\tilde{z}}^{\text{river}} = \frac{1}{\delta\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{river}} \tilde{z} + C^{\text{river}}, \quad (275)$$

$$\tilde{u}^{\text{river}} = \frac{1}{2} \frac{1}{\delta\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{river}} \tilde{z}^2 + C^{\text{river}} \tilde{z} + D^{\text{river}}, \quad (276)$$

where $C^{\text{river}}(\tilde{x})$ and $D^{\text{river}}(\tilde{x})$ are integration constants to be determined using the boundary conditions. From the boundary condition at $\tilde{z} = 0$, Eq. (273), it follows that $C^{\text{river}} = 0$. Substituting Eqs. (275) and (276) into the boundary condition at $\tilde{z} = -\tilde{H}$, Eq. (274), and using the expression for C^{river} , yields

$$D^{\text{river}} = -\frac{\tilde{H}}{\delta\lambda^2} \left(\frac{1}{\theta} + \frac{\tilde{H}}{2} \right) \tilde{\zeta}_{\tilde{x}}^{\text{river}}. \quad (277)$$

Substituting the expressions for C^{river} and D^{river} into Eq. (276) results in

$$\tilde{u}^{\text{river}} = \frac{1}{\delta\lambda^2} \left(\frac{\tilde{z}^2}{2} - \frac{\tilde{H}^2}{2} - \frac{\tilde{H}}{\theta} \right) \tilde{\zeta}_{\tilde{x}}^{\text{river}}. \quad (278)$$

The expression for $\tilde{\zeta}_{\tilde{x}}^{\text{river}}$ can be obtained by substituting Eq. (278) into the boundary condition at $\tilde{x} = 1$, Eq. (272). This leads to

$$\tilde{\zeta}_{\tilde{x}}^{\text{river}} = \frac{\delta\lambda^2 Q}{B_0 H_0 U \tilde{B} \left(\frac{\tilde{H}^3}{3} + \frac{\tilde{H}^2}{\theta} \right)}. \quad (279)$$

Numerically integrating Eq. (279) with respect to $\tilde{x}$ leads to an expression for the water level set-up $\tilde{\zeta}^{\text{river}}$ due to river outflow.

Stokes return flow: For the contribution of the Stokes return flow, the equations become

$$\tilde{u}_{\tilde{z}\tilde{z}}^{\text{stokes}} = \frac{1}{\delta\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{stokes}}, \quad (280)$$

$$\tilde{\zeta}^{\text{stokes}} = 0, \quad \text{at } \tilde{x} = 0, \quad (281)$$

$$\int_{-\tilde{H}}^0 \tilde{u}^{\text{stokes}} d\tilde{z} = -\langle \hat{\gamma} \rangle, \quad \text{at } \tilde{x} = 1, \quad (282)$$

$$\tilde{u}_{\tilde{z}}^{\text{stokes}} = 0, \quad \text{at } \tilde{z} = 0, \quad (283)$$

$$\tilde{u}_{\tilde{z}}^{\text{stokes}} = \theta \tilde{u}^{\text{stokes}}, \quad \text{at } \tilde{z} = -\tilde{H}. \quad (284)$$

Integrating Eq. (280) twice leads to

$$\tilde{u}_{\tilde{z}}^{\text{stokes}} = \frac{1}{\delta\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{stokes}} \tilde{z} + C^{\text{stokes}}, \quad (285)$$

$$\tilde{u}^{\text{stokes}} = \frac{1}{2} \frac{1}{\delta\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{stokes}} \tilde{z}^2 + C^{\text{stokes}} \tilde{z} + D^{\text{stokes}}, \quad (286)$$

where $C^{\text{stokes}}(\tilde{x})$ and $D^{\text{stokes}}(\tilde{x})$ are integration constants to be determined using the boundary conditions. From the boundary condition at $\tilde{z} = 0$, Eq. (283), it follows that $C^{\text{stokes}} = 0$. Substituting Eqs. (285) and (286) into the boundary condition at $\tilde{z} = -\tilde{H}$, Eq. (284), and using the expression for C^{stokes} , yields

$$D^{\text{stokes}} = -\frac{\tilde{H}}{\delta\lambda^2} \left(\frac{1}{\theta} + \frac{\tilde{H}}{2} \right) \tilde{\zeta}_{\tilde{x}}^{\text{stokes}}. \quad (287)$$

Substituting the expressions for C^{stokes} and D^{stokes} into Eq. (286) results in

$$\tilde{u}^{\text{stokes}} = \frac{1}{\delta\lambda^2} \left(\frac{\tilde{z}^2}{2} - \frac{\tilde{H}^2}{2} - \frac{\tilde{H}}{\theta} \right) \tilde{\zeta}_{\tilde{x}}^{\text{stokes}}. \quad (288)$$

The expression for $\tilde{\zeta}_{\tilde{x}}^{\text{stokes}}$ can be obtained by substituting Eq. (288) into the boundary condition at $\tilde{x} = 1$, Eq. (282). This leads to

$$\tilde{\zeta}_{\tilde{x}}^{\text{stokes}} = \frac{\delta\lambda^2 \langle \hat{\gamma} \rangle}{\left(\frac{\tilde{H}^3}{3} + \frac{\tilde{H}^2}{\theta} \right)}. \quad (289)$$

Numerically integrating Eq. (289) with respect to $\tilde{x}$ leads to an expression for the water level set-up $\tilde{\zeta}^{\text{stokes}}$ due to the landward net transport of water induced by the tide.

M₄ flow velocity

The equations for the M₄ flow are obtained by taking the M₄-component of the first order equations. The momentum equation, depth-averaged continuity equation and the appropriate boundary conditions are then given by

$$\delta \hat{u}_{\tilde{z}\tilde{z}}^{14} - 2i\hat{u}^{14} = \frac{1}{\lambda^2} \tilde{\zeta}_{\tilde{x}}^{14} + 2[\tilde{\xi}], \quad (290)$$

$$2i\tilde{\zeta}^{14} + \left(\frac{\partial}{\partial \tilde{x}} + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \right) \left(\int_{-\tilde{H}}^0 \tilde{u}^{14} d\tilde{z} + 2[\tilde{\gamma}] \right) = 0, \quad (291)$$

$$\tilde{\zeta}^{14} = \frac{A_{M_4}}{A_{M_2}} e^{-i\varphi}, \quad \text{at } \tilde{x} = 0, \quad (292)$$

$$\int_{-\tilde{H}}^0 \tilde{u}^{14} d\tilde{z} = -2[\tilde{\gamma}], \quad \text{at } \tilde{x} = 1, \quad (293)$$

$$\hat{u}_{\tilde{z}}^{14} = -2[\tilde{\chi}], \quad \text{at } \tilde{z} = 0, \quad (294)$$

$$\hat{u}_{\tilde{z}}^{14} = \theta \tilde{u}^{14}, \quad \text{at } \tilde{z} = -\tilde{H}. \quad (295)$$

Similar to the residual velocity, the solution to $\hat{u}^{14}$ is linear and can therefore be constructed by adding the contributions of the different forcing terms, i.e.

$$\hat{u}^{14} = \tilde{u}^{\text{tide}} + \tilde{u}^{\text{adv}} + \tilde{u}^{\text{no-stress}} + \tilde{u}^{\text{stokes}} \quad (296)$$

Notice, that the salinity field and the river outflow have no M_4 component and that an additional forcing due to the externally imposed M_4 -tide is present. To derive the expressions for the different M_4 velocity contributions only the appropriate terms due to the specific forcing are taken into account. Since an additional inertia term ($2i\hat{u}^{14}$) is present in the momentum equation (290), solving it requires the same approach as was done for the M_2 flow velocity. In the following, the different velocity contributions are derived.

M_4 external tide The solution for the M_4 flow velocity is derived in the same way as was done for the M_2 flow velocity in Section A1.2.1. The only difference is that the forcing frequency is different, i.e. $e^{2i\tilde{t}}$ instead of $e^{i\tilde{t}}$. The equations for the M_4 tide are given by

$$\delta \tilde{u}_{\tilde{z}\tilde{z}}^{\text{tide}} - 2i\tilde{u}^{\text{tide}} = \frac{1}{\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{tide}}, \quad (297)$$

$$\tilde{\zeta}^{\text{tide}} = \frac{A_{M_4}}{A_{M_2}} e^{-i\varphi}, \quad \text{at } \tilde{x} = 0, \quad (298)$$

$$\int_{-\tilde{H}}^0 \tilde{u}^{\text{tide}} d\tilde{z} = 0, \quad \text{at } \tilde{x} = 1, \quad (299)$$

$$\tilde{u}_{\tilde{z}}^{\text{tide}} = 0, \quad \text{at } \tilde{z} = 0, \quad (300)$$

$$\tilde{u}_{\tilde{z}}^{\text{tide}} = \theta \tilde{u}^{\text{tide}}, \quad \text{at } \tilde{z} = -\tilde{H}. \quad (301)$$

Following the same derivation steps as in Section A1.2.1, the solution for the M_4 flow velocity amplitude is

$$\tilde{u}^{\text{tide}} = \frac{1}{2i\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{tide}} (\tilde{\alpha}_{M_4} \cosh(\tilde{r}_{M_4} \tilde{z}) - 1), \quad (302)$$

with

$$\tilde{\alpha}_{M_4} = \frac{\theta}{\tilde{r}_{M_4} \sinh(\tilde{r}_{M_4} \tilde{H}) + \theta \cosh(\tilde{r}_{M_4} \tilde{H})}, \quad (303)$$

$$r_{M_4} = \sqrt{\frac{2i}{\delta}}. \quad (304)$$

To obtain the expression for $\tilde{\zeta}_{\tilde{x}}^{\text{tide}}$, Eq. (302) is substituted in the depth-averaged continuity equation (291). This leads to the following second-order linear homogeneous ODE for the water level

$$\tilde{F}_1 \tilde{\zeta}_{\tilde{x}\tilde{x}}^{\text{tide}} + \tilde{F}_2 \tilde{\zeta}_{\tilde{x}}^{\text{tide}} - 4\lambda^2 \tilde{\zeta}^{\text{tide}} = 0, \quad (305)$$

with

$$\tilde{F}_1 = \frac{\tilde{\alpha}_{M_4}}{\tilde{r}_{M_4}} \sinh(\tilde{r}_{M_4} \tilde{H}) - \tilde{H}, \quad (306)$$

$$\tilde{F}_2 = \left(\frac{\tilde{\alpha}_{M_4} \tilde{r}_{M_4} \tilde{H}}{\tilde{r}_{M_4}} + \tilde{\alpha}_{M_4} \tilde{H}_{\tilde{x}} \right) \cosh(\tilde{r}_{M_4} \tilde{H}) - \frac{1}{\tilde{r}_{M_4}} \left(-\frac{\tilde{\alpha}_{M_4} \tilde{B}_{\tilde{x}}}{\tilde{B}} - \tilde{\alpha}_{M_4, \tilde{x}} + \frac{\tilde{\alpha}_{M_4} \tilde{r}_{M_4, \tilde{x}}}{\tilde{r}_{M_4}} \right) \sinh(\tilde{r}_{M_4} \tilde{H}) \quad (307)$$

$$- \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \tilde{H} - \tilde{H}_{\tilde{x}}.$$

Notice that Eq. (305) is equal to Eq. (188) for the M_2 water level, i.e. same coefficients $\tilde{T}_1$ and $\tilde{T}_2$, except for a factor 4 in front of $\tilde{\zeta}^{\text{tide}}$ due to the higher frequency of the M_4 tide. Eq. (305) needs to be solved numerically since the variables $\tilde{\alpha}_{M_4}$, $\tilde{r}_{M_4}$ and $\tilde{H}$ are functions of the longitudinal coordinate $\tilde{x}$.

Advection of momentum: The equations when the contribution of the advection of momentum is considered, become

$$\delta \tilde{u}_{\tilde{z}\tilde{z}}^{\text{adv}} - 2i \tilde{u}^{\text{adv}} = \frac{1}{\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{adv}} + 2[\tilde{\xi}], \quad (308)$$

$$\tilde{\zeta}^{\text{adv}} = 0, \quad \text{at } \tilde{x} = 0, \quad (309)$$

$$\int_0^0 \tilde{u}^{\text{adv}} d\tilde{z} = 0, \quad \text{at } \tilde{x} = 1, \quad (310)$$

$$\tilde{u}_{\tilde{z}}^{\text{adv}} = 0, \quad \text{at } \tilde{z} = 0, \quad (311)$$

$$\tilde{u}_{\tilde{z}}^{\text{adv}} = \theta \tilde{u}^{\text{adv}}, \quad \text{at } \tilde{z} = -\tilde{H}. \quad (312)$$

Deriving the expression for $\tilde{u}^{\text{adv}}$, we can add the solutions to the momentum equation (308) due to the different forcing terms on the right-hand side of that equation, i.e.

$$\tilde{u}^{\text{adv}} = \tilde{u}_h^{\text{adv}} + \tilde{u}_{p, \zeta}^{\text{adv}} + \tilde{u}_{p, \xi}^{\text{adv}}, \quad (313)$$

where $\tilde{u}_h^{\text{adv}}$ is the solution of the homogeneous equation, $\tilde{u}_{p,\zeta}^{\text{adv}}$ the solution due to the water level forcing and $\tilde{u}_{p,\xi}^{\text{adv}}$ the solution due to the advection of momentum. The steps to find a solution for the first two contributions was already done for the leading order system in Section A1.2.1, and read for the homogeneous solution

$$\tilde{u}_h^{\text{adv}} = C_1 e^{\tilde{r}_{M_4} \tilde{z}} + C_2 e^{-\tilde{r}_{M_4} \tilde{z}}, \quad (314)$$

with unknown factors C_1 and C_2 . The particular solution for the water level forcing is given by

$$\tilde{u}_{p,\zeta}^{\text{adv}} = -\frac{1}{2i\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{adv}}, \quad (315)$$

Next, the solution due to the advection of momentum is found using the method of variation of parameters. This method was already introduced in the section on the contribution of advection of momentum to the residual flow velocity. What the method boils down to is finding a solution that has a similar form as the solution to the homogeneous equation, but with different factors in front of the functions $f(\tilde{x}, \tilde{z}) = e^{\tilde{r}_{M_4} \tilde{z}}$ and $g(\tilde{x}, \tilde{z}) = e^{-\tilde{r}_{M_4} \tilde{z}}$. Hence, we seek functions $A(\tilde{x}, \tilde{z})$ and $B(\tilde{x}, \tilde{z})$, such that

$$\tilde{u}_{p,\xi}^{\text{adv}} = Af + Bg, \quad (316)$$

is a general solution of the non-homogeneous equation. We need only to calculate the integrals

$$A(\tilde{x}, \tilde{z}) = - \int_{-\tilde{H}}^{\tilde{z}} \frac{1}{W} g(\tilde{x}, \tilde{z}') b(\tilde{x}, \tilde{z}') d\tilde{z}', \quad (317)$$

$$B(\tilde{x}, \tilde{z}) = \int_{-\tilde{H}}^{\tilde{z}} \frac{1}{W} f(\tilde{x}, \tilde{z}') b(\tilde{x}, \tilde{z}') d\tilde{z}', \quad (318)$$

where W is the Wronskian of the functions f and g given by

$$W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} e^{\tilde{r}_{M_4} \tilde{z}} & e^{-\tilde{r}_{M_4} \tilde{z}} \\ \tilde{r}_{M_4} e^{\tilde{r}_{M_4} \tilde{z}} & -\tilde{r}_{M_4} e^{-\tilde{r}_{M_4} \tilde{z}} \end{vmatrix} = -\tilde{r}_{M_4} e^{-\tilde{r}_{M_4} \tilde{z}} e^{\tilde{r}_{M_4} \tilde{z}} - \tilde{r}_{M_4} e^{-\tilde{r}_{M_4} \tilde{z}} e^{\tilde{r}_{M_4} \tilde{z}} = -2\tilde{r}_{M_4}. \quad (319)$$

Furthermore, $b(\tilde{x}, \tilde{z}) = 2[\hat{\xi}]$ is the non-homogeneous forcing term. Substituting W , f , g and b in Eqs. (317) and (318), results in

$$A(\tilde{x}, \tilde{z}) = \frac{1}{\tilde{r}_{M_4}} \int_{-\tilde{H}}^{\tilde{z}} [\hat{\xi}] e^{-\tilde{r}_{M_4} \tilde{z}'} d\tilde{z}', \quad (320)$$

$$B(\tilde{x}, \tilde{z}) = -\frac{1}{\tilde{r}_{M_4}} \int_{-\tilde{H}}^{\tilde{z}} [\hat{\xi}] e^{\tilde{r}_{M_4} \tilde{z}'} d\tilde{z}', \quad (321)$$

The final general solution to Eq. (241) can be found by adding Eqs. (314), (315) and (316) and reads

$$\tilde{u}^{\text{adv}} = C_1 e^{\tilde{r}_{M_4} \tilde{z}} + C_2 e^{-\tilde{r}_{M_4} \tilde{z}} + \frac{1}{\tilde{r}_{M_4}} G_1 - \frac{1}{2i\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{adv}}, \quad (322)$$

with

$$G_1 = e^{\tilde{r}_{M_4} \tilde{z}} \int_{-\tilde{H}}^{\tilde{z}} [\hat{\xi}] e^{-\tilde{r}_{M_4} \tilde{z}'} d\tilde{z}' - e^{-\tilde{r}_{M_4} \tilde{z}} \int_{-\tilde{H}}^{\tilde{z}} [\hat{\xi}] e^{\tilde{r}_{M_4} \tilde{z}'} d\tilde{z}'. \quad (323)$$

The unknowns C_1 and C_2 are obtained using the boundary conditions at the free surface, Eq. (311), and at the bottom, Eq. (312). At the free surface, $\tilde{u}_{\tilde{z}}^{\text{adv}} = 0$. Taking the derivative of Eq. (322) with respect to $\tilde{z}$ and equalling to zero results in

$$C_1 \tilde{r}_{M_4} - C_2 \tilde{r}_{M_4} + \frac{1}{\tilde{r}_{M_4}} \left(\left[\hat{\xi} \right]_{\tilde{z}=0} + \tilde{r}_{M_4} \int_{-\tilde{H}}^0 [\hat{\xi}] e^{-\tilde{r}_{M_4} \tilde{z}'} d\tilde{z}' - \left[\hat{\xi} \right]_{\tilde{z}=0} + \tilde{r}_{M_4} \int_{-\tilde{H}}^0 [\hat{\xi}] e^{\tilde{r}_{M_4} \tilde{z}'} d\tilde{z}' \right) = 0. \quad (324)$$

Rearranging this equation leads to

$$C_2 = C_1 + \frac{1}{\tilde{r}_{M_4}} G_2, \quad (325)$$

with

$$G_2 = \int_{-\tilde{H}}^0 [\hat{\xi}] e^{-\tilde{r}_{M_4} \tilde{z}'} d\tilde{z}' + \int_{-\tilde{H}}^0 [\hat{\xi}] e^{\tilde{r}_{M_4} \tilde{z}'} d\tilde{z}'. \quad (326)$$

At the bottom, $\tilde{u}_{\tilde{z}}^{\text{adv}} = \theta \tilde{u}^{\text{adv}}$. Substituting the expression for $\tilde{u}^{\text{adv}}$ into Eq. (312) and rearranging results in

$$C_1 = -\frac{1}{2} \frac{\tilde{\alpha}_{M_4}}{\theta} \left(\left(1 + \frac{\theta}{\tilde{r}_{M_4}} \right) G_2 e^{\tilde{r}_{M_4} \tilde{H}} - \frac{1}{2i\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{adv}} \right). \quad (327)$$

Substituting the expressions for C_1 and C_2 into Eq. (322) gives

$$\tilde{u}^{\text{adv}} = \frac{1}{2i\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{adv}} (\tilde{\alpha} \cosh(\tilde{r}_{M_4} \tilde{z}) - 1) + \frac{G_1}{\tilde{r}_{M_4}} - \frac{\tilde{\alpha}_{M_4} G_2}{\theta \tilde{r}_{M_4}} \left(\tilde{r}_{M_4} \cosh(\tilde{r}_{M_4} (\tilde{z} + \tilde{H})) + \theta \sinh(\tilde{r}_{M_4} (\tilde{z} + \tilde{H})) \right). \quad (328)$$

To obtain the expression for $\tilde{\zeta}_{\tilde{x}}^{\text{adv}}$, Eq. (328) is substituted into the depth-averaged continuity equation (291). This leads to the following second-order linear non-homogeneous ODE for the water level

$$\tilde{F}_1 \tilde{\zeta}_{\tilde{x}\tilde{x}}^{\text{adv}} + \tilde{F}_2 \tilde{\zeta}_{\tilde{x}}^{\text{adv}} - 4\lambda^2 \tilde{\zeta}^{\text{adv}} = \tilde{F}_{\text{adv}}, \quad (329)$$

with

$$\tilde{F}_{\text{adv}} = \left(\frac{d}{dx} + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \right) \left(\frac{G_2}{\tilde{r}_{M_4}^2} (1 - \tilde{\alpha}_{M_4}) - \frac{1}{\tilde{r}_{M_4}} \int_{-\tilde{H}}^0 G_1 d\tilde{z} \right), \quad (330)$$

and $\tilde{F}_1$ and $\tilde{F}_2$ defined by Eqs. (306) and (307), respectively. Eq. (329) needs to be solved numerically since the variables $\tilde{\alpha}_{M_4}$, $\tilde{r}_{M_4}$ and $\tilde{H}$ are functions of the longitudinal coordinate $\tilde{x}$.

Stress free boundary condition: The equations due to the contribution of the stress free boundary condition are given by

$$\delta \tilde{u}_{\tilde{z}\tilde{z}}^{\text{no-stress}} - 2i\tilde{u}^{\text{no-stress}} = \frac{1}{\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{no-stress}}, \quad (331)$$

$$\tilde{\zeta}^{\text{no-stress}} = 0, \quad \text{at } \tilde{x} = 0, \quad (332)$$

$$\int_{-\tilde{H}}^0 \tilde{u}^{\text{no-stress}} d\tilde{z} = 0, \quad \text{at } \tilde{x} = 1, \quad (333)$$

$$\tilde{u}_{\tilde{z}}^{\text{no-stress}} = -2[\hat{\chi}], \quad \text{at } \tilde{z} = 0, \quad (334)$$

$$\tilde{u}_{\tilde{z}}^{\text{no-stress}} = \theta \tilde{u}^{\text{no-stress}}, \quad \text{at } \tilde{z} = -\tilde{H}. \quad (335)$$

The general solution to the momentum equation (331) is given by the contribution due to the homogeneous equation and the water level forcing. This solution is obtained by adding Eqs. (314) and (315) and reads

$$\tilde{u}^{\text{no-stress}} = C_1 e^{\tilde{r}_{M_4} \tilde{z}} + C_2 e^{-\tilde{r}_{M_4} \tilde{z}} - \frac{1}{2i\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{no-stress}}. \quad (336)$$

The unknowns C_1 and C_2 are obtained using the boundary conditions at the free surface and the bottom. From the boundary condition at the free surface, Eq. (334), it follows that

$$C_2 = C_1 + \frac{2[\hat{\chi}]}{\tilde{r}_{M_4}}. \quad (337)$$

Subsequently, it follows from the bottom boundary condition, Eq. (335), that

$$C_1 = \frac{\tilde{\alpha}_{M_4}}{\theta} \left(\frac{\theta}{4i\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{no-stress}} - \left(1 + \frac{\theta}{\tilde{r}_{M_4}} \right) [\hat{\chi}] e^{\tilde{r}_{M_4} \tilde{H}} \right). \quad (338)$$

Substituting the expressions for C_1 and C_2 , Eqs. (338) and (337), respectively, into Eq. (336) results in

$$\tilde{u}^{\text{no-stress}} = \frac{1}{2i\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{no-stress}} (\tilde{\alpha}_{M_4} \cosh(\tilde{r}_{M_4} \tilde{z}) - 1) - \frac{2\tilde{\alpha}_{M_4}}{\theta \tilde{r}_{M_4}} [\hat{\chi}] \left(\tilde{r}_{M_4} \cosh(\tilde{r}_{M_4} (\tilde{z} + \tilde{H})) + \theta \sinh(\tilde{r}_{M_4} (\tilde{z} + \tilde{H})) \right). \quad (339)$$

To obtain the expression for $\tilde{\zeta}_{\tilde{x}}^{\text{no-stress}}$, Eq. (339) is substituted into the depth-averaged continuity equation (291). This leads to the following second-order linear non-homogeneous ODE for the water level

$$\tilde{F}_1 \tilde{\zeta}_{\tilde{x}\tilde{x}}^{\text{no-stress}} + \tilde{F}_2 \tilde{\zeta}_{\tilde{x}}^{\text{no-stress}} - 4\lambda^2 \tilde{\zeta}^{\text{no-stress}} = \tilde{F}_{\text{no-stress}}, \quad (340)$$

with

$$\tilde{F}_{\text{no-stress}} = \left(\frac{d}{dx} + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \right) \left(\frac{2[\hat{\chi}]}{\tilde{r}_{M_4}} (1 - \tilde{\alpha}_{M_4}) \right), \quad (341)$$

and $\tilde{F}_1$ and $\tilde{F}_2$ defined by Eqs. (306) and (307), respectively. Eq. (340) needs to be solved numerically since the variables $\tilde{\alpha}_{M_4}$, $\tilde{r}_{M_4}$ and $\tilde{H}$ are functions of the longitudinal coordinate $\tilde{x}$.

Stokes return flow: The equations due to the contribution of the Stokes return flow are given by

$$\delta \tilde{u}_{\tilde{z}\tilde{z}}^{\text{stokes}} - 2i \tilde{u}^{\text{stokes}} = \frac{1}{\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{stokes}}, \quad (342)$$

$$\tilde{\zeta}^{\text{stokes}} = 0, \quad \text{at } \tilde{x} = 0, \quad (343)$$

$$\int_0^{\tilde{H}} \tilde{u}^{\text{stokes}} d\tilde{z} = -2[\hat{\gamma}], \quad \text{at } \tilde{x} = 1, \quad (344)$$

$$\tilde{u}_{\tilde{z}}^{\text{stokes}} = 0, \quad \text{at } \tilde{z} = 0, \quad (345)$$

$$\tilde{u}_{\tilde{z}}^{\text{stokes}} = \theta \tilde{u}^{\text{stokes}}, \quad \text{at } \tilde{z} = -\tilde{H}. \quad (346)$$

The general solution to the momentum equation (342) is given by the contribution due to the homogeneous equation and the water level forcing. This solution is equal to Eq. (268) and reads

$$\tilde{u}^{\text{stokes}} = C_1 e^{\tilde{r}_{M_4} \tilde{z}} + C_2 e^{-\tilde{r}_{M_4} \tilde{z}} - \frac{1}{2i\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{stokes}}. \quad (347)$$

The unknowns C_1 and C_2 are obtained using the boundary conditions at the free surface and the bottom. From the boundary condition at the free surface, Eq. (345), it follows that $C_1 = C_2 = C$.

Subsequently, it follows from the bottom boundary condition, Eq. (346), that

$$C = \frac{\tilde{\alpha}_{M_4}}{4i\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{stokes}}. \quad (348)$$

Substituting the expression for C , Eq. (348), into Eq. (347) results in

$$\tilde{u}^{\text{stokes}} = \frac{1}{2i\lambda^2} \tilde{\zeta}_{\tilde{x}}^{\text{stokes}} (\tilde{\alpha}_{M_4} \cosh(\tilde{r}_{M_4} \tilde{z}) - 1). \quad (349)$$

To obtain the expression for $\tilde{\zeta}_{\tilde{x}}^{\text{stokes}}$, Eq. (349) is substituted into the depth-averaged continuity equation (291). This leads to the following second-order linear non-homogeneous ODE for the water level

$$\tilde{F}_1 \tilde{\zeta}_{\tilde{x}\tilde{x}}^{\text{stokes}} + \tilde{F}_2 \tilde{\zeta}_{\tilde{x}}^{\text{stokes}} - 4\lambda^2 \tilde{\zeta}^{\text{stokes}} = \tilde{F}_{\text{stokes}}, \quad (350)$$

with

$$\tilde{F}_{\text{stokes}} = - \left(\frac{d}{dx} + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}} \right) 2[\hat{\gamma}], \quad (351)$$

and $\tilde{F}_1$ and $\tilde{F}_2$ defined by Eqs. (306) and (307), respectively. Eq. (350) needs to be solved numerically since the variables $\tilde{\alpha}_{M_4}$, $\tilde{r}_{M_4}$ and $\tilde{H}$ are functions of the longitudinal coordinate $\tilde{x}$.

A2 Along-channel velocity and least square harmonic fit

The observed along-channel depth averaged velocity $u^{\text{obs}}(x, t)$ was used to determine the harmonic constituents of the horizontal tide. To this end, the temporal velocity was approximated by a series $u^{\text{aPP}}(x, t)$ that includes only residual, M_2 and M_4 components, i.e.

$$u^{\text{aPP}}(x, t) = \mathcal{A}_{M0} + \mathcal{A}_{M2} \cos(\sigma t) + \mathcal{B}_{M2} \sin(\sigma t) + \mathcal{A}_{M4} \cos(2\sigma t) + \mathcal{B}_{M4} \sin(2\sigma t) . \quad (352)$$

Next the harmonic components $\mathcal{A}_{M0}, \dots, \mathcal{B}_{M4}$ were determined for a location at position x by minimization of the quantity

$$\Delta = \sum_{i=1}^N [u^{\text{obs}}(x, t_i) - u^{\text{aPP}}(x, t_i)]^2 ,$$

where $\{t_i\}$ denotes the set of times for which ADCP measurements are available. The minimization leads to a linear matrix problem that is typical of least square approximations.

The values for the harmonic components thus obtained can be converted to amplitude and phase information. For instance, the M_2 amplitude $\bar{u}_{M2}$ and phase ϕ_{M2} follow from

$$\bar{u}_{M2} = \sqrt{\mathcal{A}_{M2}^2 + \mathcal{B}_{M2}^2} , \quad \tan \phi_{M2} = \frac{\mathcal{B}_{M2}}{\mathcal{A}_{M2}} .$$

Figures 23 and 24 show, for each location in Table 7, the measured temporal along-channel velocities and the harmonic approximation according to Eq. (352) above. Here positive and negative velocities correspond to horizontal flood and ebb, respectively.

Figure 23 – Comparison between measured depth averaged along-estuary velocity (red) and least square fit according to Eq. (352) (blue) for the six most seaward locations.

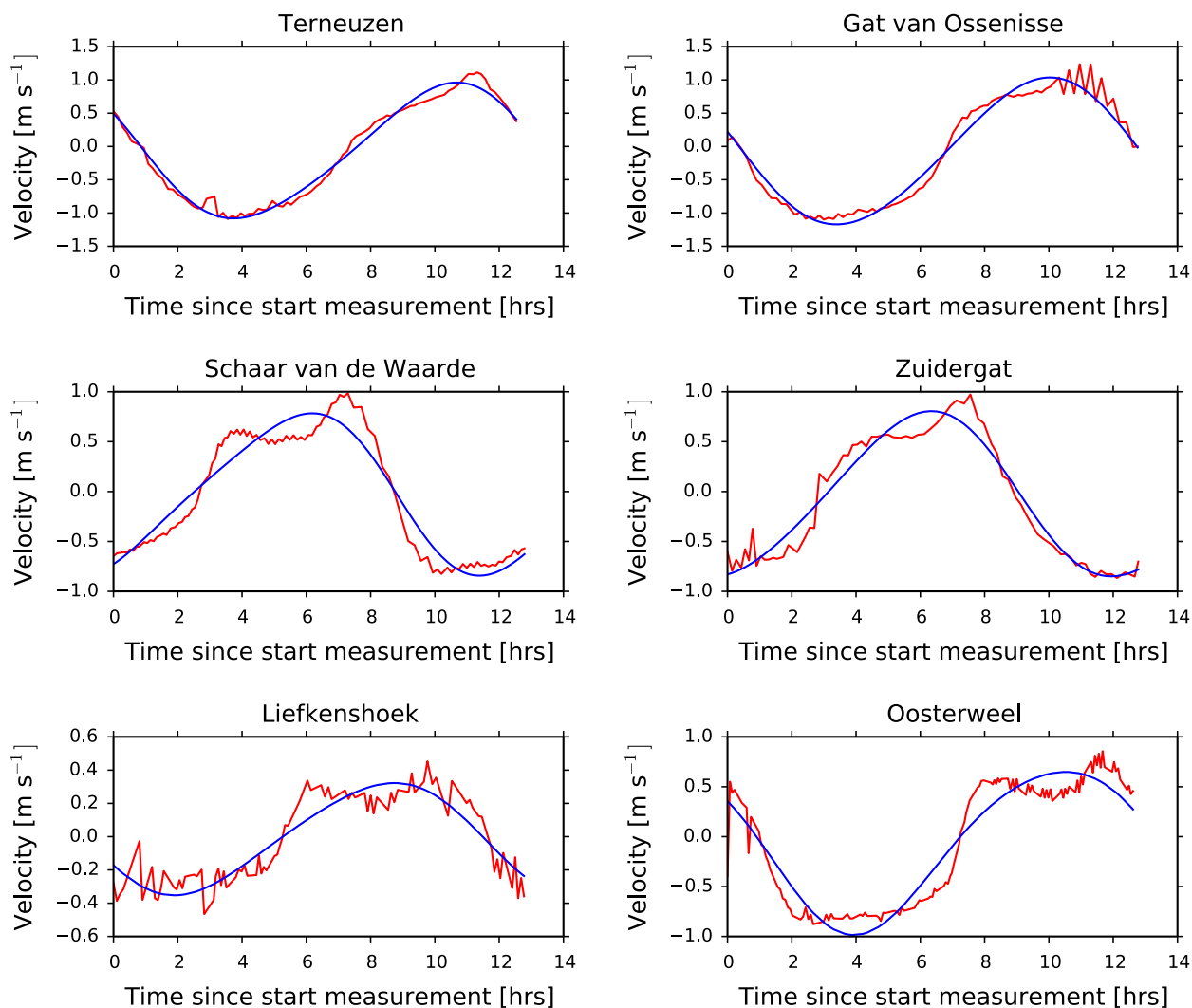
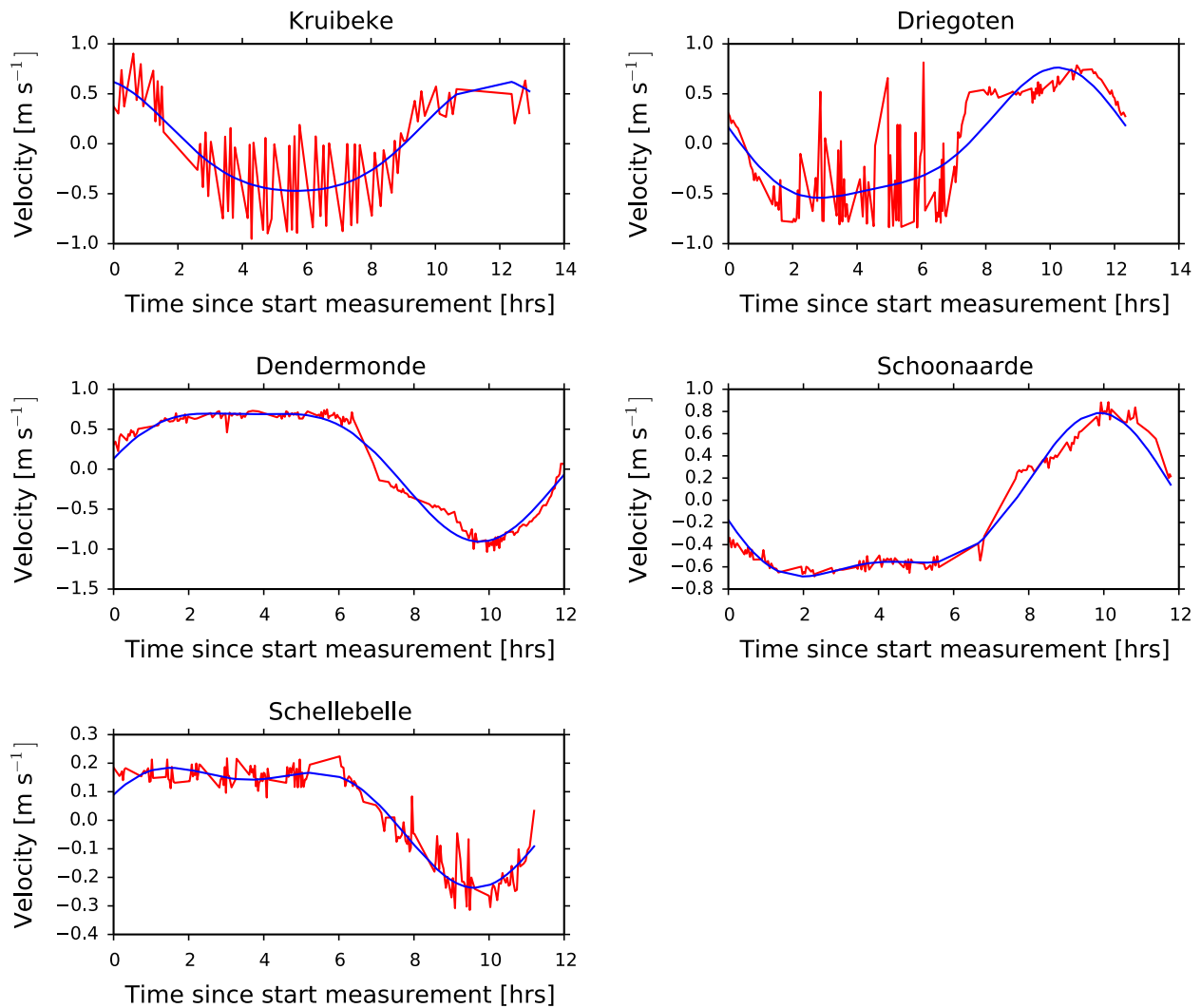


Figure 24 – Comparison between measured depth averaged along-estuary velocity (red) and least square fit according to Eq. (352) (blue) for the five most landward locations.



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